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On oscillation of solutions of nonlinear neutral hyperbolic equations with delay

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This paper establishes the existence of some sufficient conditions for oscillation of nonlinear neutral hyperbolic equations with delay. Our results generalize and extend those reported in the literature. Examples are included to illustrate the importance of the results obtained. Copyright © 2025 Shahid Beheshti University.

Keywords: Oscillation; Neutral hyperbolic equation; Delay.

1. Introduction

There is much interest in studying the oscillatory behavior of solutions of hyperbolic equations. Our objective here is to provide oscillation theorems for nonlinear neutral hyperbolic equations with delay of the form

$$\frac{\partial}{\partial t}\Big(r(t)\frac{\partial}{\partial t}\Big(u(x,t)+h(t)u(x,\tau(t))\Big)\Big)+p(t)\frac{\partial}{\partial t}u(x,\rho(t))-a(t)\Delta u(x,t)-b(t)\Delta u(x,\eta(t)) + q(x,t)\varphi\Big(u(x,\sigma(t))\Big)=f(x,t),\ (x,t)\in G\times(0,\infty)\equiv\Omega \tag{E$_+$}$$

and

$$\frac{\partial}{\partial t}\Big(r(t)\frac{\partial}{\partial t}\Big(u(x,t)+h(t)u(x,\tau(t))\Big)\Big)-p(t)\frac{\partial}{\partial t}u(x,\rho(t))-a(t)\Delta u(x,t)-b(t)\Delta u(x,\eta(t)) + q(x,t)\varphi\Big(u(x,\sigma(t))\Big)=f(x,t),\ (x,t)\in \Omega, \tag{E$_-$}$$

where G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G and Δ is the Laplacian in $\mathbb{R}^n$, with one of the following boundary conditions

$$u=0,\ (x,t)\in\partial G\times(0,\infty), \tag{B1}$$

$$\frac{\partial u}{\partial \nu}+\gamma(x,t)u=0,\ (x,t)\in\partial G\times(0,\infty), \tag{B2}$$

where $\gamma(x,t)$ is a non-negative continuous function on $\partial G\times(0,\infty)$ and ν denotes the unit exterior normal vector to ∂G.

We assume throughout this paper that the following conditions hold:

(H1) $r(t),h(t),p(t),a(t),b(t)\in C([0,\infty];[0,\infty))$;
$\tau(t),\eta(t),\sigma(t)\in C([0,\infty];[0,\infty))$ $\rho(t)\in C^2([0,\infty];[0,\infty))$;
$\tau(t)\le t,\ \rho(t)\le t,\ \sigma(t)\le t$;
$\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\rho(t)=\lim_{t\to\infty}\eta(t)=\lim_{t\to\infty}\sigma(t)=\infty$;

(H2) $q(x,t)\in C(\bar{\Omega};[0,\infty))$;
$\varphi(\xi)\in C(\mathbb{R};\mathbb{R})$ are convex in $[0,\infty)$ and $\xi\varphi(\xi)>0$ for $\xi\neq 0$;

(H3) $f(x,t)\in C(\bar{\Omega};\mathbb{R})$.

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Definition 1 *By a solution of Eq.* ($E_\pm$) *we mean a function* $u \in C^2(\bar{G} \times [t_{-1}, \infty)) \times C^1(\bar{G} \times [\hat{t}_{-1}, \infty) \times C(\bar{G} \times [\tilde{t}_{-1}, \infty)$ *which satisfies Eq.* ($E_\pm$), *where*

$$t_{-1} = \min\left\{0, \inf_{t \geq 0}(\tau(t))\right\}, \quad \hat{t}_{-1} = \min\left\{0, \inf_{t \geq 0}(\rho(t))\right\},$$
$$\tilde{t}_{-1} = \min\left\{0, \inf_{t \geq 0}(\sigma(t))\right\}.$$

Definition 2 *A solution* u *of Eq.* ($E_\pm$) *is said to be oscillatory in* Ω *if* u *has a zero in* $G \times (T, \infty)$ *for any* $T > 0$.

Introduce the following notation:

$$q(t) = \min\{q(x,t) : x \in \bar{G}\},$$
$$\pi(t) = \int_t^\infty \frac{1}{r(s)} ds.$$

With each solution u of the problems ($E_\pm$), (B1) or ($E_\pm$), (B2) we associate the function

$$U(t) \equiv K_\Phi \int_G u(x,t)\Phi(x)dx, \quad \tilde{U}(t) \equiv \frac{1}{|G|}\int_G u(x,t)dx,$$
$$F(t) \equiv K_\Phi \int_G f(x,t)\Phi(x)dx, \quad \tilde{F}(t) \equiv \frac{1}{|G|}\int_G f(x,t)dx,$$

where $K_\Phi = (\int_G \Phi(x)dx)^{-1}$ and $|G| = \int_G dx$.

It divides into the following two cases (P1) and (P2):

$$\text{(P1)} \int_{t_0}^\infty \frac{1}{r(s)} ds = \infty, \qquad \text{(P2)} \int_{t_0}^\infty \frac{1}{r(s)} ds < \infty.$$

In recent years, there has been a great deal of interest in studying oscillatory behavior of solutions to nonlinear fractional and differential equations; see [2], [3], [4], [5]. The problem of establishing oscillation criteria for boundary value problems for hyperbolic equations has been reported by several authors [6], [7], [8] [10], [11] [1], [12], [14], [13], [15], [9]], and the references cited therein. However, to the author's knowledge, the problem represented by hyperbolic equation (E) with constant coefficients has studied at only one paper. This corresponding results for hyperbolic equation (E) is due to Kirane and Parhi, see [8], but we did not find a paper which gave sufficient conditions to guarantee the existence of an oscillatory solutions for hyperbolic equation (E). On the basis of the ideas exploited by Tanaka [14], Feng and Sun [7], we derive some renew oscillation results for nonlinear neutral hyperbolic equation (E). As we will explain in the examples, we can see that the results of this paper can be used to solve problems that cannot be solved by the results of [14].

2. Reduction to one-dimensional problems

In this section we reduce the multi-dimensional problems for ($E_\pm$) to one-dimensional problems. When we investigate the oscillatory properties of the solution of problem ($E_\pm$), (B1) , we use following eigenvalue problem. It is known that the first eigenvalue λ_1 of the eigenvalue problem

$$-\Delta w = \lambda w \quad \text{in} \quad G,$$
$$w = 0 \quad \text{on} \quad \partial G$$

is positive, and the corresponding eigenfunction $\Phi(x)$ can be chosen so that $\Phi(x) > 0$ in G.

Theorem 1 *If the functional differential inequalities*

$$(r(t)(x(t) + h(t)x(\tau(t)))')' + p(t)x'(\rho(t)) + q(t)\varphi(x(\sigma(t))) \leq \pm F(t) \tag{1}$$

have no eventually positive solution, then every solution $u(x,t)$ *of the problem* (E_+), (B1) *is oscillatory in* Ω.

Proof. Suppose to the contrary that there is a nonoscillatory solution u of the problem (E_+), $(\mathrm{B1})$. Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$, because the case where $u(x,t) < 0$ can be treated similarly. Since (H1) holds, we see that $u(x, \tau(t)) > 0$, $u(x, \rho(t)) > 0$, $u(x, \eta(t)) > 0$ and $u(x, \sigma(t)) > 0$ in $G \times [t_1, \infty)$ for some $t_1 \geq t_0$. Multiplying (E_+) by $\Phi(x)K_\Phi$ and integrating over G, we obtain

$$\begin{aligned} &(r(t)(U(t) + h(t)U(\tau(t)))')' + p(t)U'(\rho(t)) \\ &- a(t)K_\Phi \int_G \Delta u(x,t)\Phi(x)dx - b(t)K_\Phi \int_G \Delta u(x,\eta(t))\Phi(x)dx \\ &+ K_\Phi \int_G q(x,t)\varphi(u(x,\sigma(t)))\Phi(x)dx = F(t), \;\; t \geq t_1. \end{aligned} \tag{2}$$

From Green's formula it follows that

$$K_\Phi \int_G \Delta u(x,t)\Phi(x)dx = -\lambda_1 U(t) \leq 0, \;\; t \geq t_1, \tag{3}$$

$$K_\Phi \int_G \Delta u(x,\eta(t))\Phi(x)dx = -\lambda_1 U(\eta(t)) \leq 0, \;\; t \geq t_1. \tag{4}$$

An application of Jensen's inequality shows that

$$K_\Phi \int_G q(x,t)\varphi(u(x,\sigma(t)))\Phi(x)dx \geq q(t)\varphi(U(\sigma(t))), \;\; t \geq t_1. \tag{5}$$

Combining (2) – (5) yields

$$(r(t)(U(t) + h(t)U(\tau(t)))')' + p(t)U'(\rho(t)) + q(t)\varphi(U(\sigma(t))) \leq F(t), \;\; t \geq t_1.$$

Therefore $U(t)$ is an eventually positive solution of (1). This contradicts the hypothesis and copletes the proof. □

Similarly we can prove the problem (E_-), $(\mathrm{B1})$, and so, we omit the following proof of theorem.

Theorem 2 *If the functional differential inequalities*

$$(r(t)(x(t) + h(t)x(\tau(t)))')' - p(t)x'(\rho(t)) + q(t)\varphi(x(\sigma(t))) \leq \pm\tilde{F}(t) \tag{6}$$

have no eventually positive solution, then every solution $u(x,t)$ of the problem (E_-), $(\mathrm{B1})$ is oscillatory in Ω.

Theorem 3 *If the functional differential inequalities (1) have no eventually positive solution, then every solution $u(x,t)$ of the problem (E_+), $(\mathrm{B2})$ is oscillatory in Ω.*

Proof. Suppose to the contrary that there is a nonoscillatory solution u of the problem (E_+), $(\mathrm{B2})$. Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Since (H1) holds, we see that $u(x, \tau(t)) > 0$, $u(x, \rho(t)) > 0$, $u(x, \eta(t)) > 0$ and $u(x, \sigma(t)) > 0$ in $G \times [t_1, \infty)$ for some $t_1 \geq t_0$. Dividing (E_+) by $|G|$ and integrating over G, we obtain

$$\begin{aligned} &(r(t)(\tilde{U}(t) + h(t)\tilde{U}(\tau(t)))')' + p(t)\tilde{U}'(\rho(t)) \\ &- \frac{a(t)}{|G|} \int_G \Delta u(x,t)dx - \frac{b(t)}{|G|} \int_G \Delta u(x,\eta(t))dx + \frac{1}{|G|} \int_G q(x,t)\varphi(u(x,\sigma(t)))dx = \tilde{F}(t), \;\; t \geq t_1. \end{aligned} \tag{7}$$

It follows from Green's formula that

$$\int_G \Delta u(x,t)dx = \int_{\partial G} \frac{\partial u}{\partial \nu}(x,t)dS = -\int_{\partial G} \gamma(x,t)u(x,t)dS \leq 0, \tag{8}$$

$$\int_G \Delta u(x,\eta(t))dx = \int_{\partial G} \frac{\partial u}{\partial \nu}(x,\eta(t))dS = -\int_{\partial G} \gamma(x,\eta(t))u(x,\eta(t))dS \leq 0 \tag{9}$$

for $t \geq t_1$. Applying Jensen's inequality, we observe that

$$\frac{1}{|G|} \int_G q(x,t)\varphi(u(x,\sigma(t)))dx \geq q(t)\varphi(\tilde{U}(\sigma(t))), \;\; t \geq t_1. \tag{10}$$

Combining (7) – (10) yields

$$(r(t)(\tilde{U}(t) + h(t)\tilde{U}(\tau(t)))')' + p(t)\tilde{U}'(\rho(t)) + q(t)\varphi(\tilde{U}(\sigma(t))) \leq \tilde{F}(t), \;\; t \geq t_1.$$

Therefore $\tilde{U}(t)$ is an eventually positive solution of (1). This contradicts the hypothesis and complete the proof. □

Theorem 4 *If the functional differential inequalities (6) have no eventually positive solution, then every solution $u(x,t)$ of the problem (E_-), $(\mathrm{B2})$ is oscillatory in Ω.*

3. Nonlinear differential inequalities of neutral type

From the discussion in Section 2 it follows that the problem of establishing oscillation criteria for the Eq. ($E_\pm$) can be reduced to the investigation of the properties of the solution of second order neutral differential inequalities. The following notation will be used:

$$\Theta(t) = \theta(t) - h(t)\theta(\tau(t)) - \int_{t_0}^{t} \frac{p(s)\theta(\rho(s))}{r(s)\rho'(s)} ds,$$
$$\tilde{\Theta}(t) = \theta(t) - h(t)\theta(\tau(t)) - \int_{t}^{\infty} \frac{p(s)\theta(\rho(s))}{r(s)\rho'(s)} ds,$$
$$[\varphi(z)]_+ = \max\{\varphi(z), 0\}.$$

Theorem 5 *Under the hypothesis* (P1) *holds and the following*:

(H4) $\left(\frac{p(t)}{\rho'(t)}\right)' \le 0$;

(H5) *there exists a function* $\theta \in C^2[t_0,\infty)$ *such that* $r\theta \in C^1[t_0,\infty)$, $(r(t)\theta'(t))' = f(t)$ *and* $\theta(t)$ *is oscillatory.*

If

$$\int_{t_0}^{\infty} q(t)\varphi\left(\left[\left(1 - h(\sigma(t)) - \int_{t_0}^{\sigma(t)} \frac{p(s)}{r(s)\rho'(s)} ds\right) c + \Theta(\sigma(t))\right]_+\right) dt = \infty \tag{11}$$

for every $c > 0$, *then the differential inequality*

$$(r(t)(y(t) + h(t)y(\tau(t)))')' + p(t)y'(\rho(t)) + q(t)\varphi(y(\sigma(t))) \le f(t) \tag{12}$$

has no eventually positive solutions.

Proof. Suppose that this is not true and let $y(t)$ be a positive solution of inequality (12) in $[t_1,\infty)$, where $t_1 \ge t_0$. We set

$$z(t) = y(t) + h(t)y(\tau(t)) + \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} y(\rho(s)) ds - \theta(t), \tag{13}$$

then it is easy to check that

$$(r(t)z'(t))' - \left(\frac{p(t)}{\rho'(t)}\right)' y(\rho(t)) + q(t)\varphi(y(\sigma(t))) \le 0,\ \ t \ge t_1. \tag{14}$$

From (H4) we see that

$$(r(t)z'(t))' \le 0. \tag{15}$$

Hence we conclude that $r(t)z'(t)$ is nonincreasing for $t \ge t_1$, so that $z'(t) \ge 0$ or $z'(t) < 0$ for $t \ge t_1$. We can find that $z(t) > 0$ for $t \ge t_1$. In fact if $z(t) \le 0$ for $t \ge t_1$, then

$$0 \ge z(t) \ge y(t) - \theta(t),$$

which contradicts that $\theta(t)$ is oscillatory. Next if we suppose that $z'(t) < 0$ for $t \ge t_1$. Integrating (15) over $[t_1, t]$ yields

$$r(t)z'(t) \le r(t_0)z'(t_0) < 0, t \ge t_1. \tag{16}$$

Multiplying (16) by $1/r(t)$ and integrating over $[t_1, t]$, we obtain

$$z(t) \le z(t_0) + r(t_0)z'(t_0) \int_{t_1}^{t} \frac{1}{r(s)} ds,\ \ t \ge t_1.$$

Letting $t \to \infty$ in above inequality, we show that $\lim_{t\to\infty} z(t) = -\infty$ because of (P1). This contradicts the fact that $z(t) > 0$ for $t \ge t_1$, and so, $z(t) > 0$ and $z'(t) \ge 0$ for $t \ge t_1$. In view of (13), we see that

$$y(t) = z(t) - h(t)y(\tau(t)) - \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} y(\rho(s)) ds + \theta(t). \tag{17}$$

On the other hand, we have

$$y(t) \le z(t) + \theta(t) \tag{18}$$

for $t \geq t_1$. Substituting (18) into (17) yields

$$y(t) \geq z(t) - h(t)\Big(z(\tau(t)) + \theta(\tau(t))\Big) - \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)}\Big(z(\rho(s)) + \theta(\rho(s))\Big)ds + \theta(t)$$
$$= z(t) - h(t)z(\tau(t)) - \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} z(\rho(s))ds + \Theta(t),\ t \geq t_1.$$

Since $z(t) > 0$ and $z'(t) \geq 0$, we show that

$$z(t) \geq c,\ t \geq t_1$$

for every $c > 0$. From the above discussion it is obvious that

$$y(t) \geq \left(1 - h(t) - \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} ds\right) c + \Theta(t),\ t \geq t_1.$$

Since $y(t)$ is eventually positive, we have

$$y(t) \geq \left[\left(1 - h(t) - \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} ds\right) c + \Theta(t)\right]_+ \equiv H(t),\ t \geq t_1. \tag{19}$$

Take $t_2 \geq t_1$ so large that $\sigma(t) \leq t_1$ for $t \geq t_2$. Combining (14) and (19) we have

$$(r(t)z'(t))' + q(t)\varphi\left(H(\sigma(t))\right) \leq 0,\ t \geq t_1.$$

Integrating the above inequality over $[t_2, t]$ yields

$$\int_{t_2}^{t} q(s)\varphi\left(H(\sigma(s))\right) ds \leq r(t_2)z'(t_2) - r(t)z'(t) \leq r(t_2)z'(t_2),$$

which implies that

$$\int_{t_2}^{\infty} q(s)\varphi\left(H(\sigma(s))\right) ds < \infty.$$

This contradicts the condition (11). □

Theorem 6 *Under the hypothesis* (P1), (H5) *hold and the following*:

(H6) $\left(\frac{p(t)}{\rho'(t)}\right)' \geq 0.$

If

$$\int_{t_0}^{\infty} q(t)\varphi\left(\left[\left(1 - h(\sigma(t)) - \int_{\sigma(t)}^{\infty} \frac{p(s)}{r(s)\rho'(s)} ds\right) c + \tilde{\Theta}(\sigma(t))\right]_+\right) dt = \infty \tag{20}$$

for every $c > 0$, then the differential inequality

$$(r(t)(y(t) + h(t)y(\tau(t)))')' - p(t)y'(\rho(t)) + q(t)\varphi(y(\sigma(t))) \leq f(t) \tag{21}$$

has no eventually positive solutions.

Proof. Suppose that this is not true and let $y(t)$ be a positive solution of inequality (12) in $[t_1, \infty)$, where $t_1 \geq t_0$. Setting

$$w(t) = y(t) + h(t)y(\tau(t)) + \int_{t}^{\infty} \frac{p(s)}{r(s)\rho'(s)} y(\rho(s))ds - \theta(t), \tag{22}$$

then it is easy to check that

$$(r(t)w'(t))' + \left(\frac{p(t)}{\rho'(t)}\right)' y(\rho(t)) + q(t)\varphi(y(\sigma(t))) \leq 0,\ t \geq t_1. \tag{23}$$

From (H6) we see that

$$(r(t)w'(t))' \leq 0.$$

As in the proof of Theorem 5 it is easy to see that $w(t) > 0$ and $w'(t) \geq 0$ for $t \geq t_1$. Also we can lead to the following inequality

$$y(t) \geq \left[\left(1 - h(t) - \int_{t}^{\infty} \frac{p(s)}{r(s)\rho'(s)} ds\right) c + \Theta(t)\right]_+ \equiv \tilde{H}(t),\ t \geq t_1. \tag{24}$$

Combining (23) and (24) we have

$$(r(t)w'(t))' + q(t)\varphi\left(\tilde{H}(\sigma(t))\right) \le 0, \ \ t \ge t_1.$$

Integrating the above inequality over $[t_2, t]$ yields

$$\int_{t_2}^{\infty} q(s)\varphi\left(\tilde{H}(\sigma(s))\right) ds < \infty.$$

This contradicts the condition (20). □

Next we need the following lemmas by Tanaka[12] for the case of (P2).

Lemma 1 *Suppose that* (P2) *hold. If* $y(t)$ *is an eventually positive solution of* (21), *then* $z(t)$ *is bounded above and satisfies*

$$z(t) \ge -\pi(t)r(t)z'(t) \tag{25}$$

for all sufficiently large $t \ge t_0$.

Lemma 2 *Assume that* (P2) *hold. Let* $y(t)$ *be an eventually positive solution of* (21). *Then there exist a constant* $c > 0$ *and a number* $t_2 \ge t_1$ *such that*

$$z(t) \ge c\pi(t), \ \ t \ge t_2.$$

Theorem 7 *Under the hypothesis* (P2), (H4) *and* (H5) *hold. If*

$$\int_{t_0}^{\infty} \pi(t)q(t)\varphi\left([cA(\sigma(t))\pi(\gamma(\sigma(t))) + \Theta(\sigma(t))]_+\right) dt = \infty \tag{26}$$

for every $c > 0$, *then the differential inequality* (12) *has no eventually positive solutions, where*

$$A(t) = \left(\frac{\pi(t)}{\pi(\gamma(t))} - h(t) - \int_{t_0}^{t} \frac{p(s)}{r(s)\rho'(s)} ds\right),$$
$$\gamma(t) = \min\{\tau(t), \rho(t), t\}.$$

Proof. If differential inequality (12) has an eventually positive solution, then there exists $t_1 \ge t_0$ such that $y(t) > 0$ for $t \ge t_1$. As in the proof of Theorem 5, we obtain

$$(r(t)z'(t))' \le 0, \ \ t \ge t_1.$$

Then it follows that either

$$z(t) > 0, \ z'(t) \ge 0 \qquad \text{or} \qquad z(t) > 0, \ z'(t) < 0$$

holds for $t \ge t_1$. Now we assume that $z(t) > 0$, $z'(t) \ge 0$ holds. Since $\pi(t)$ is decreasing, we obtain

$$y(t) \ge \left(1 - \left(h(t) + \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} ds\right) \frac{\pi(\gamma(t))}{\pi(t)}\right) z(t) + \Theta(t), \ \ t \ge t_1. \tag{27}$$

Next we assume that $z(t) > 0$, $z'(t) < 0$ holds. From the definition of $z(t)$ we obtain

$$y(t) = z(t) - h(t)z(\tau(t)) - z(\rho(\xi)) \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} ds + \Theta(t), \ \ t \ge t_1$$

for some $\xi \in [t_1, t]$. It follows from Lemma 1 that (25) hold. Then it is obvious that $\left(\frac{z(t)}{\pi(t)}\right)$ is nondecreasing (see, [7]), since

$$\left(\frac{z(t)}{\pi(t)}\right)' = \frac{z'(t)\pi(t) - z(t)\pi'(t)}{\pi^2(t)} = \frac{r(t)z'(t)\pi(t) + z(t)}{\pi^2(t)} \ge 0$$

for $t \ge t_1$. Hence, it is easy to see that

$$y(t) \ge \left(1 - h(t)\left(\frac{\pi(\tau(t))}{\pi(t)}\right) - \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} ds \left(\frac{\pi(\rho(\xi))}{\pi(t)}\right)\right) z(t) + \Theta(t) \tag{28}$$

for $t \ge t_1$. Combining Lemma 2 with (27) or (28) respectively, we have

$$y(t) \ge c\left(\pi(t) - \left(h(t) + \int_{t_1}^{t} \frac{p(s)}{r(s)\rho'(s)} ds\right) \pi(\gamma(t))\right) + \Theta(t), \ \ t \ge t_1.$$

Substituting this inequality into (14) yields

$$(r(t)z'(t))' + q(t)\varphi([cA(\sigma(t))\pi(\gamma(\sigma(t))) + \Theta(\sigma(t))]_+) \le 0, \ \ t \ge t_1. \tag{29}$$

Multiplying (29) by $\pi(t)$ and integrating over $[t_1, t]$, we obtain

$$\begin{aligned} &\pi(t)r(t)z'(t) + z(t) - \pi(t_1)r(t_1)z'(t_1) - z(t_1) \\ &\qquad + \int_{t_1}^{t} \pi(s)q(s)\varphi([cA(\sigma(s))\pi(\gamma(\sigma(s))) + \Theta(\sigma(s))]_+)ds \le 0. \end{aligned}$$

Using the relation (25), the above inequality reduces to

$$\int_{t_1}^{t} \pi(s)q(s)\varphi([cA(\sigma(s))\pi(\gamma(\sigma(s))) + \Theta(\sigma(s))]_+)ds \le \pi(t_1)r(t_1)z'(t_1) + z(t_1),$$

which we can leads to the contradiction and completes the proof. □

Corollary 1 *Under the hypothesis* (P2) *and* (H4) *hold. If*

$$\liminf_{t\to\infty} \int_{t_0}^{t} (\pi(t) - \pi(s))f(s)ds = -\infty, \tag{30}$$

then the differential inequality (12) *has no eventually positive solutions.*

Proof. Let $y(t)$ be an eventually positive solution of the differential inequality (12). Then there exists $t_1 \ge t_0$ such that $y(t) > 0$ for $t \ge t_1$. We choose the following function

$$\tilde{z}(t) = y(t) + h(t)y(\tau(t)) + \int_{t_0}^{t} \frac{p(s)}{r(s)\rho'(s)} y(\rho(s))ds.$$

Then we rewrite (12) as

$$(r(t)\tilde{z}'(t))' \le f(t), \ \ t \ge t_1.$$

Integrating the above inequality over $[t_1, t]$ and dividing both sides by $r(t)$, we have

$$\tilde{z}'(t) \le r(t_1)\tilde{z}'(t_1)/r(t) + \frac{1}{r(t)} \int_{t_1}^{t} f(s)ds.$$

Further integration yields

$$\tilde{z}(t) \le \tilde{z}(t_1) + r(t_1)\tilde{z}'(t_1) \int_{t_1}^{t} \frac{1}{r(s)}ds + \int_{t_1}^{t} \frac{1}{r(s)} \int^{s} f(u)duds.$$

Taking inferior limit as $t \to \infty$, it is clear that

$$0 \le \liminf_{t\to\infty} \tilde{z}(t) \le -\infty,$$

which contradicts and completes the proof. □

Theorem 8 *Under the hypothesis* (P2), (H5) *and* (H6) *hold. If*

$$\int_{t_0}^{\infty} \pi(t)q(t)\varphi\left(\left[c\tilde{A}(\sigma(t))\pi(\gamma(\sigma(t))) + \tilde{\Theta}(\sigma(t))\right]_+\right) dt = \infty \tag{31}$$

for every $c > 0$, then the differential inequality (21) *has no eventually positive solutions, where*

$$\tilde{A}(t) = \left(\frac{\pi(t)}{\pi(\gamma(t))} - h(t) - \int_{t}^{\infty} \frac{p(s)}{r(s)\rho'(s)}ds\right).$$

Proof. Let $y(t)$ be an eventually positive solution of the differential inequality (21). Then there exists $t_1 \geq t_0$ such that $y(t) > 0$ for $t \geq t_1$. From the function $z(t)$ as in (22) it follows that

$$y(t) = z(t) - h(t)y(\tau(t)) - \int_t^\infty \frac{p(s)}{r(s)\rho'(s)} y(\rho(s))ds + \theta(t) \tag{32}$$

for $t \geq t_1$. Substituting $z(t) \geq y(t) - \theta(t)$ into (32) yields

$$y(t) = z(t) - h(t)z(\tau(t)) - \int_t^\infty \frac{p(s)}{r(s)\rho'(s)} z(\rho(s))ds + \tilde{\Theta}(t),$$

which implies that

$$y(t) = z(t) - h(t)z(\tau(t)) - z(\rho(\xi)) \int_t^\infty \frac{p(s)}{r(s)\rho'(s)} ds + \tilde{\Theta}(t)$$

for some $\xi \in [t, \infty]$. Using the same proof of theorem 7 we have

$$y(t) \geq \left[c\tilde{A}(t)\pi(\gamma(t)) + \tilde{\Theta}(t)\right]_+, \ t \geq t_1. \tag{33}$$

Hence it follows from (23) that

$$(r(t)z'(t))' + q(t)\varphi\left(\left[c\tilde{A}(\sigma(t))\pi(\gamma(\sigma(t))) + \tilde{\Theta}(\sigma(t))\right]_+\right) \leq 0, \ t \geq t_1.$$

By integrating the above inequality over $[t_1, t]$ and letting as $t \to \infty$, we can lead to the contradiction. □

Corollary 2 *Under the hypothesis* (P2) *and* (H6) *hold. If the condition* (30) *holds, then the differential inequality* (21) *has no eventually positive solutions.*

Proof. Let $y(t)$ be an eventually positive solution of the differential inequality (12). Then there exists $t_1 \geq t_0$ such that $y(t) > 0$ for $t \geq t_1$. We define the following function:

$$\tilde{w}(t) = y(t) + h(t)y(\tau(t)) + \int_t^\infty \frac{p(s)}{r(s)\rho'(s)} y(\rho(s))ds$$

for $t \geq t_1$. Substituting it into (12) we obtain

$$(r(t)\tilde{w}'(t))' \leq f(t), \ t \geq t_1.$$

By a similar proof of Theorem 7 we can show that

$$\tilde{w}(t) \leq \tilde{w}(t_1) + r(t_1)\tilde{w}'(t_1) \int_{t_1}^t \frac{1}{r(s)} ds + \int_{t_1}^t \frac{1}{r(s)} \int^s f(u)duds.$$

Taking inferior limit as $t \to \infty$, it is easy to see that

$$0 \leq \liminf_{t\to\infty} \tilde{w}(t) \leq -\infty,$$

which contradicts and completes the proof. □

4. Oscillation criteria for the Eq. ($\mathrm{E}_\pm$) and examples

In this section we derive oscillation criteria for the equation ($\mathrm{E}_\pm$). Combining Theorems 1–8 and Corollaries 1–2, we establish following oscillation results under the hypothesis:

(H7) there exists a function $\theta \in C^2[t_0, \infty)$ such that $r\theta \in C^1[t_0, \infty)$, $(r(t)\theta'(t))' = F(t)$ and $\theta(t)$ is oscillatory;
(H8) there exists a function $\theta \in C^2[t_0, \infty)$ such that $r\theta \in C^1[t_0, \infty)$, $(r(t)\theta'(t))' = \tilde{F}(t)$ and $\theta(t)$ is oscillatory.

Finally, we discuss some examples which dwell upon the importance of our main results are given.

Theorem 9 *Assume that* (P1), (H4) *and* (H7) (*or* (H8)) *hold. If conditions*

$$\int_{t_0}^\infty q(t)\varphi\left(\left[\left(1 - h(\sigma(t)) - \int_{t_0}^{\sigma(t)} \frac{p(s)}{r(s)} ds\right) c \pm \Theta(\sigma(t))\right]_+\right) dt = \infty$$

holds, then every solution (E_+), (B1) (*or* (E_+), (B2)) *is oscillatory in* Ω.

Theorem 10 *Assume that* (P1) *and* (H5), (H7) (*or* (H8)) *hold. If conditions*

$$\int_{t_0}^{\infty} q(t)\varphi\left(\left[\left(1-h(\sigma(t))-\int_{\sigma(t)}^{\infty}\frac{p(s)}{r(s)}ds\right)c\pm\tilde{\Theta}(\sigma(t))\right]_+\right)dt=\infty$$

holds, then every solution (E_-), (B1) (*or* (E_-), (B2)) *is oscillatory in* Ω.

Theorem 11 *Assume that* (P2), (H4) *and* (H7) (*or* (H8)) *hold. If conditions*

$$\int_{t_0}^{\infty}\pi(t)q(t)\varphi\left([cA(\sigma(t))\pi(\gamma(\sigma(t)))\pm\Theta(\sigma(t))]_+\right)dt=\infty$$

holds, then every solution (E_+), (B1) (*or* (E_+), (B2)) *is oscillatory in* Ω.

Corollary 3 *Assume that* (P2) *and* (H4) *hold. If conditions*

$$\liminf_{t\to\infty}\int_{t_0}^{t}(\pi(t)-\pi(s))F(s)ds=-\infty, \tag{34}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}(\pi(t)-\pi(s))F(s)ds=\infty \tag{35}$$

hold, then every solution (E_+), (B1) *is oscillatory in* Ω.

Corollary 4 *Assume that* (P2) *and* (H4) *hold. If conditions*

$$\liminf_{t\to\infty}\int_{t_0}^{t}(\pi(t)-\pi(s))\tilde{F}(s)ds=-\infty, \tag{36}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}(\pi(t)-\pi(s))\tilde{F}(s)ds=\infty \tag{37}$$

hold, then every solution (E_+), (B2) *is oscillatory in* Ω.

Theorem 12 *Assume that* (P2) *and* (H5), (H7) (*or* (H8)) *hold. If conditions*

$$\int_{t_0}^{\infty}\pi(t)q(t)\varphi\left(\left[c\tilde{A}(\sigma(t))\pi(\gamma(\sigma(t)))\pm\tilde{\Theta}(\sigma(t))\right]_+\right)dt=\infty$$

holds, then every solution (E_-), (B1) (*or* (E_-), (B2)) *is oscillatory in* Ω.

Corollary 5 *Assume that* (P2) *and* (H6) *hold. If conditions* (34) *and* (35) *hold, then every solution* (E_-), (B1) *is oscillatory in* Ω.

Corollary 6 *Assume that* (P2) *and* (H6) *hold. If conditions* (36) *and* (37) *hold, then every solution* (E_-), (B2) *is oscillatory in* Ω.

This section concludes with some examples illustrating the main results stated above.

Example 1 *Consider the problem*

$$\frac{\partial^2}{\partial t^2}\left(u(x,t)+\frac{1}{4}e^{-t}u(x,t-2\pi)\right)+\frac{1}{2}e^{-t}\frac{\partial}{\partial t}u(x,t-3\pi/2)$$
$$-\Delta u(x,t)+\frac{1}{3}u(x,t-\pi)=-\frac{1}{3}\sin x\cos t,\ \ (x,t)\in(0,\pi)\times(t_0,\infty) \tag{38}$$

with boundary condition

$$u(0,t)=u(\pi,t)=0,\ \ t\geq 0. \tag{39}$$

In this case $r(t)=1$, $h(t)=\frac{1}{4}e^{-t}$, $p(t)=\frac{1}{2}e^{-t}$, $q(x,t)=\frac{5}{3}$, $\tau(t)=t-2\pi$, $\rho(t)=t-3\pi/2$, $\sigma(t)=t-\pi$, $f(x,t)=-\frac{1}{3}\sin x\cos t$. *It is not difficult to see that the conditions of Theorem 9 are satisfied, since*

$$\int^{\infty}\frac{1}{3}\left[\left(1+\frac{1}{4}e^{-t+\pi}-\frac{1}{2}e^{-t_0+\pi}\right)c\pm\Theta(\sigma(t))\right]_+dt=\infty,$$

where

$$\Theta(\sigma(t))=\frac{\pi}{12}\left(-\cos t+\frac{1}{4}e^{-t+\pi}\sin t+\frac{1}{2}e^{-t+\pi}\cos t+\frac{1}{4}e^{-t_0}\cos t_0+\frac{1}{4}e^{-t_0}\sin t_0\right)$$

Therefore every solutions of the problem (38), (39) oscillates in $(0,\pi)\times(t_0,\infty)$. *For instance,* $u(x,t)=\sin x\cos t$ *is such a solution.*

Example 2 *Consider the problem*

$$\frac{\partial}{\partial t}\left(e^t\frac{\partial}{\partial t}\left(u(x,t)+hu(x,t-\pi)\right)\right)-\frac{\partial}{\partial t}u(x,t)-e^{2t}\Delta u(x,t)-\sqrt{2}(1-h)e^t\Delta u(x,t-\pi/2) \quad (40)$$
$$+e^{2t}u(x,t-\pi)=-\sin x\cos t,\ (x,t)\in(0,\pi)\times(1,\infty)$$

with boundary condition

$$u(0,t)=u(\pi,t)=0,\ t\geq 0, \quad (41)$$

where $r(t)=e^t$, $h=e^{-\pi}-e^{-\varepsilon}$, $p(t)=1$, $q(x,t)=e^{2t}$, $\tau(t)=t-\pi$, $\rho(t)=t$, $\sigma(t)=t-2\pi$, $f(x,t)=-\sin x\cos t$. *Obviously, the conditions of Theorem 12 are satisfied, since*

$$\int^{\infty} e^t\left[c\left(e^{-\pi}-h-e^{-t}\right)e^{-t+\pi}\pm\tilde{\Theta}(\sigma(t))\right]_+=\infty$$

where

$$\tilde{\Theta}(\sigma(t))=\frac{\pi}{8}\left\{-(1+he^{\pi})(\cos t-\sin t)e^{-t+\pi}+e^{-2t+2\pi}\left(\frac{1}{5}\cos t+\frac{3}{5}\sin t\right)\right\}$$

for some $t_0>0$. *But result of Tanaka [14] is not applicable to this problem since*

$$1-h-\log\left(\frac{\pi(\tau(t))}{\pi(t)}\right)=1-h-\pi<0.$$

Therefore every solutions of problem (40), (41) oscillates in $(0,\pi)\times(t_0,\infty)$. *In fact* $u(x,t)=\sin x\sin t$ *is such a solution.*

Example 3 *Consider the problem*

$$\frac{\partial}{\partial t}\left(t^2\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{t}u(x,t-2\pi)\right)\right)-\frac{\partial}{\partial t}u(x,t-3\pi/2)-t\Delta u(x,t)-2t\Delta u(x,t-\pi/2)$$
$$+t^2u(x,t-2\pi)=\sin x\sin t,\ (x,t)\in(0,\pi)\times(1,\infty)$$

with boundary condition

$$u(0,t)=u(\pi,t)=0,\ t\geq 0, \quad (42)$$

where $r(t)=t^2$, $h(t)=1/t$, $p(t)=1$, $q(x,t)=t^2$, $\tau(t)=t-2\pi$, $\rho(t)=t$, $\sigma(t)=t-2\pi$, $f(x,t)=-\sin x\sin t$. *Obviously, the conditions of Theorem 12 are satisfied, since*

$$\int^{\infty} t\left[c\left(1-\frac{2\pi+2}{t-2\pi}\right)\left(\frac{1}{t-2\pi}\right)\pm\tilde{\Theta}(\sigma(t))\right]_+=\infty$$

where

$$\tilde{\Theta}(\sigma(t))=-\frac{\pi}{4}\int_{t_0}^{t-2\pi}\frac{\cos s}{s^2}ds+\frac{\pi/4}{t-4\pi}\int_{t_0}^{t-2\pi}\frac{\cos s}{s^2}ds+\frac{\pi}{4}\int_{t-2\pi}^{\infty}\frac{1}{s^2}\int_{s_0}^{s-3\pi/2}\frac{\cos\xi}{\xi^2}d\xi ds$$

for some $t_0>0$. *But Corollary 3 is not applicable to this problem since*

$$\lim_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{t}-\frac{1}{s}\right)\cdot\frac{\pi}{4}\sin s ds=-\frac{\pi^2}{8}.$$

*Therefore every solutions of problem (**??**), (42) oscillates in* $(0,\pi)\times(t_0,\infty)$. *In fact* $u(x,t)=\sin x\sin t$ *is such a solution.*

5. Conclusions

In previous results [1], Parhi and Chand established established some sufficient conditions under which every bounded solution of linear neutral hyperbolic equations with delay either oscillates or tends to zero as $t\to\infty$. Therefore we showed oscillation theorems for nonlinear neutral hyperbolic equations with delay.

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