Research Article ISSN Online: 2783-4859 **Computational Mathematics and Computer Modeling with Applications**

Received 2 August 2024
Accepted 26 September 2024

DOI: 10.48308/CMCMA.3.1.38
AMS Subject Classification: 34B15; 65L60

A fully discretization approach for nonlinear Phi-four equations

Mohammad Ali Mehrpouya[a]

In this paper, a fully discretization approach is established for accurate and efficient solution of nonlinear time-dependent Phi-four equations arising in particle physics and quantum mechanics. In the suggested approach, the lobatto pseudospectral method is used to discretize the desired problem. So, the Phi-four equation is converted into a set of nonlinear algebraic equations. The primary benefit of the suggested approach is that, it produces excellent results with just few discretization points and has a fast rate of convergence. Numerical results are showcased to verify the precision and effectiveness of the suggested approach for solving nonlinear Phi-four equations.

Keywords: Phi-four equations; Discretization methods; Collocation; Legendre-Gauss-Lobatto points.

1. Introduction

Numerous problems in the examination of theoretical physics stem from the famous nonlinear Klein-Gordon equation, which is structured as

$$u_{tt}(x,t) = u_{xx}(x,t) + h(u(x,t)) + s(x,t),$$

where, $u(x,t)$ denotes the wave displacement at position x and time t, while $h(u(x,t))$ stands for a nonlinear force. In the present paper, we are going to study a special and best-known case of the Klein-Gordon equation called the nonlinear Phi-four equation. Actually, we deal with the numerical approximation of the nonlinear time-dependent Phi-four equation, which has the form

$$u_{tt}(x,t) = \lambda_1 u_{xx}(x,t) + \lambda_2 u(x,t) + \lambda_3 u^{\alpha}(x,t) + s(x,t), \quad (x,t) \in \Omega \times [0,T], \tag{1}$$

where, λ_1, λ_2, λ_3 and α are constants and $\Omega = \{x : A \leq x \leq B\}$. The associated boundary and initial conditions are respectively given by

$$u(A,t) = f_1(t), \;\; u(B,t) = f_2(t), \tag{2}$$

$$u(x,0) = g_1(x), \;\; u_t(x,0) = g_2(x), \;\; x \in \Omega. \tag{3}$$

The nonlinear Phi-four equations have been used to describe diverse phenomena, such as the interaction between kink and antikink solitary waves in particle physics [2] or the model phenomenon in relativistic quantum mechanics [4]. There have been few methods in the literature to address the Phi-four equations. Chowdhury and Biswas [4] approximated the singular soliton solution of the Phi-four equation using the ansatz method. This approach converts the nonlinear Phi-four equation, a partial differential equation, into a set of ordinary differential equations (ODEs) in time using a spectral method whose basis functions are rational Chebyshev functions. Then, the obtained system of ODEs can be solved using any standard numerical ODE integration algorithm. In a similar manner, Bhrawy et al. [2] transformed the solution of nonlinear Phi-four equation to the solution of a set of ODEs in time using an orthogonal collocation method based on the Jacobi polynomials. In [9], various explicit and implicit finite difference methods of fourth and sixth orders are presented to address the Phi-four equation. In [32] and [31], a cubic and trigonometric B-spline collocation methods are utilized respectively for numerical solution of the Phi-four equation. Arora and Bhatia [1] presented a computational method which is based on radial basis function pseudospectral method. Their approach involves employing radial basis functions to discretize the space derivatives in the Phi-four equation. Next, the Phi-four equation is converted into a set of ODEs that can be solved using an ODE-solver. Recently Mehboob Ul Haq et al. [12] developed a

[a] *Department of Mathematics, Tafresh University, Tafresh 39518-79611, Iran.*
[*] *Correspondence to: M.A. Mehrpouya. Email: mehrpouya@tafreshu.ac.ir*

numerical scheme based on using non-polynomial spline for the solution of the Phi-four equation. However, in addition to the mentioned numerical methods, some analytical and semi-analytical methods have been reported to solve some special variants of the Phi-four equation, which we can refer the interested readers to Wazwaz and Triki [30], Najafi [25], Demiray and Bulut [8] and khater et al. [15].

The goal of this study is to employ the lobatto pseudospectral method to solve the nonlinear Phi-four equation numerically. Pseudospectral methods are advanced tools for solving ordinary and partial differential equations [3, 11, 7, 14]. It is worth mentioning that, the Lobatto pseudospectral method utilizes the Legendre-Gauss-Lobatto (LGL) points in its application as an orthogonal collocation technique. Many applications of this type of pseudospectral methods have been reported in solving partial differential equations. For instances, we can refer the interested readers to the works done in [19, 17, 16, 18, 29, 5, 28]. In cases where a smooth problem and its solution need to be accurately solved on a simple domain, the pseudospectral methods are typically the most effective tool. The characteristic feature of these methods is that, with a small number of discretization points, they succeed in achieving very high accuracies.

2. The proposed method

In this section, we introduce the lobatto pseudospectral method for solving the nonlinear Phi-four equation (1), while also considering the boundary and initial conditions specified in Eqs. (2) and (3). The proposed method consists of two parts. First, it is important to think about a suitable finite depiction of the solution to equation (1). This part of the method is done with the help of polynomial interpolation of solution based on using the LGL points. The next part involves deriving a set of algebraic equations by discretizing Eqs. (1)-(3) at the collocation points.

Denote the Legendre polynomials of degree n and m as P_n and P_m, respectively. The renowned Legendre polynomials are popular polynomials defined on the interval $[-1,+1]$ and have demonstrated success in approximating functions over finite intervals [22, 24, 13, 21, 27, 20, 23]. Now, suppose that $\{\eta_i\}_{i=0}^{n}$ and $\{\theta_j\}_{j=0}^{m}$ denote the LGL points, in which $\eta_0=-1$, $\theta_0=-1$, $\eta_n=+1$, $\theta_n=+1$ and for $1<=i<=n-1$, $1<=j<=m-1$, η_i and θ_j are the zeros of $\dot{P}_n(x)=\frac{d}{dx}P_n(x)$ and $\dot{P}_m(t)=\frac{d}{dt}P_m(t)$, respectively. So, a function of two variables $\mathscr{H}(x,t):[a,b]\times[t_0,t_f]\to\mathbb{R}$ can be approximated by

$$\mathscr{H}(x,t)\simeq\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i,j}\phi_i\Big(\frac{2x-(b+a)}{b-a}\Big)\varphi_j\Big(\frac{2t-(t_f+t_0)}{t_f-t_0}\Big), \tag{4}$$

where,

$$c_{i,j}=\mathscr{H}\Big(\frac{(b-a)\eta_i+(b+a)}{2},\frac{(t_f-t_0)\theta_j+(t_f+t_0)}{2}\Big),$$

and $\phi_i(x)$, $i=0,\dots,n$, and $\varphi_j(t)$, $j=0,\dots,m$, are the Lagrange polynomials which are based on $\{\eta_i\}_{i=0}^{n}$ and $\{\theta_j\}_{j=0}^{m}$, that are established by

$$\phi_i(x)=\prod_{k=0,k\neq i}^{n}\frac{x-\eta_k}{\eta_i-\eta_k},\quad i=0,\dots,n,$$
$$\varphi_j(t)=\prod_{l=0,l\neq j}^{m}\frac{t-\theta_l}{\theta_j-\theta_l},\quad j=0,\dots,m,$$

and have the Kronecker property

$$\phi_i(\eta_k)=\begin{cases}0, & \text{if } i\neq k,\\ 1, & \text{if } i=k,\end{cases} \tag{5}$$

$$\varphi_j(\theta_l)=\begin{cases}0, & \text{if } j\neq l,\\ 1, & \text{if } j=l.\end{cases} \tag{6}$$

Now, with the knowledge of how to approximate a function of two variables in this method, we are ready to use the lobatto pseudospectral method for discretizing the problem (1)-(3). To achieve this goal, the problem's solution ($u_{ex}(x,t)$) is approximated with $u_{app}(x,t)$ using Eq. (4). So, we have

$$u_{app}(x,t)\simeq\sum_{i=0}^{n}\sum_{j=0}^{m}c_{i,j}\phi_i\Big(\frac{2x-(B+A)}{B-A}\Big)\varphi_j\Big(\frac{2}{T}t-1\Big). \tag{7}$$

It is noted that, there are $(n+1)\times(m+1)$ unknown coefficients $c_{i,j}$ in the Eq. (7), which should be obtained. Now, substituting $u_{app}(x,t)$ to the nonlinear Phi-four equation (1), we have

$$R(x,t)=\frac{\partial}{\partial t^2}u_{app}(x,t)-\lambda_1\frac{\partial}{\partial x^2}u_{app}(x,t)-\lambda_2u_{app}(x,t)-\lambda_3u_{app}^{\alpha}(x,t)-s(x,t). \tag{8}$$

Recognizing that, R is the residual function and minimizing it, is crucial. In this study, the collocation method is applied to a set of LGL points in order to make the residual vanishes pointwisely. Now, collocating the residual function $R(x,t)$ at the points $(\hat{\eta}_i, \hat{\theta}_j)$, $i = 1, \dots, n-1$, $j = 1, \dots, m-1$, where $\{\hat{\eta}_0, \hat{\eta}_1, \dots, \hat{\eta}_n\}$ and $\{\hat{\theta}_0, \hat{\theta}_1, \dots, \hat{\theta}_m\}$ are shifted LGL points to the intervals $[A, B]$ and $[0, T]$ respectively, $(n-1) \times (m-1)$ algebraic equations which is demanded to find the unknown coefficients $c_{i,j}$ is obtained as

$$\frac{\partial}{\partial t^2} u_{app}(\hat{\eta}_i, \hat{\theta}_j) - \lambda_1 \frac{\partial}{\partial x^2} u_{app}(\hat{\eta}_i, \hat{\theta}_j) - \lambda_2 u_{app}(\hat{\eta}_i, \hat{\theta}_j) - \lambda_3 u^{\alpha}_{app}(\hat{\eta}_i, \hat{\theta}_j) - s(\hat{\eta}_i, \hat{\theta}_j) = 0, \quad i = 1, \dots, n-1, \ j = 1, \dots, m-1, \tag{9}$$

and of course additional algebraic equations must be derived from the boundary and initial conditions (2) and (3). In a comparable manner, to discretize the boundary and initial conditions in Eqs. (2) and (3), we have

$$u_{app}(A, \hat{\theta}_p) - f_1(\hat{\theta}_p) = 0, \ p = 1, \dots, m-1, \tag{10}$$
$$u_{app}(B, \hat{\theta}_p) - f_2(\hat{\theta}_p) = 0, \ p = 1, \dots, m-1, \tag{11}$$
$$u_{app}(\hat{\eta}_q, 0) - g_1(\hat{\eta}_q) = 0, \ q = 1, \dots, n-1, \tag{12}$$
$$\frac{\partial}{\partial t} u_{app}(\hat{\eta}_q, 0) - g_2(\hat{\eta}_q) = 0, \ q = 1, \dots, n-1, \tag{13}$$

and therefore, $2(m-1) + 2(n-1)$ other algebraic equations are also provided to find the coefficients $c_{i,j}$. The only thing that needs to provide the remaining 4 algebraic equations. Here, using boundary conditions (2), we have

$$u_{app}(A, \hat{\theta}_0) - f_1(\hat{\theta}_0) = 0, \tag{14}$$
$$u_{app}(A, \hat{\theta}_n) - f_1(\hat{\theta}_n) = 0, \tag{15}$$
$$u_{app}(B, \hat{\theta}_0) - f_2(\hat{\theta}_0) = 0, \tag{16}$$
$$u_{app}(B, \hat{\theta}_n) - f_2(\hat{\theta}_n) = 0. \tag{17}$$

Finally, $(n-1) \times (m-1)$ equations (9) beside $2(m-1) + 2(n-1)$ equations in (10)-(13) further 4 equations in (14)-(17) form the following system of nonlinear algebraic equations

$$\begin{cases} \frac{\partial}{\partial t^2} u_{app}(\hat{\eta}_i, \hat{\theta}_j) - \lambda_1 \frac{\partial}{\partial x^2} u_{app}(\hat{\eta}_i, \hat{\theta}_j) - \lambda_2 u_{app}(\hat{\eta}_i, \hat{\theta}_j) - \lambda_3 u^{\alpha}_{app}(\hat{\eta}_i, \hat{\theta}_j) - s(\hat{\eta}_i, \hat{\theta}_j) = 0, \quad i = 1, \dots, n-1, \ j = 1, \dots, m-1, \\ u_{app}(A, \hat{\theta}_p) - f_1(\hat{\theta}_p) = 0, \ p = 1, \dots, m-1, \\ u_{app}(B, \hat{\theta}_p) - f_2(\hat{\theta}_p) = 0, \ p = 1, \dots, m-1, \\ u_{app}(\hat{\eta}_q, 0) - g_1(\hat{\eta}_q) = 0, \ q = 1, \dots, n-1, \\ \frac{\partial}{\partial t} u_{app}(\hat{\eta}_q, 0) - g_2(\hat{\eta}_q) = 0, \ q = 1, \dots, n-1, \\ u_{app}(A, \hat{\theta}_0) - f_1(\hat{\theta}_0) = 0, \\ u_{app}(A, \hat{\theta}_n) - f_1(\hat{\theta}_n) = 0, \\ u_{app}(B, \hat{\theta}_0) - f_2(\hat{\theta}_0) = 0, \\ u_{app}(B, \hat{\theta}_n) - f_2(\hat{\theta}_n) = 0. \end{cases} \tag{18}$$

After solving the systems of nonlinear algebraic equations (18), the values of unknown coefficients $\{c_{i,j}\}$ in the Eq. (7) are obtained and therefore, the solution of the nonlinear time-dependent Phi-four equation is completed. It is noted that, the Jacobian matrix of the system of nonlinear algebraic equations (18) is not sparse. Nevertheless, the absence of sparsity is not a factor the effectiveness of the proposed method. Due to our discovery that, the proposed method offers precise outcomes for small number of discretization points. As a result, the Jacobian matrix of the system of nonlinear algebraic equations (18) is not large in size and consequently sparsity in not very large dimensions is not important.

3. Numerical illustrations

In this section, two examples are provided to show the relevance and accuracy of the proposed method. Furthermore, the calculations are executed on a Core i5 PC Laptop operating at 2.53 GHz, equipped with 4GB of memory, utilizing Maple 2015 with the Digits environment set to 16. The system of nonlinear algebraic equations (18) is solved using the Maple function `fsolve`. Moreover, the absolute error function is used to depict the difference between the exact solution $u_{ex}(x,t)$ and the approximated solution $u_{app}(x,t)$ as following

$$E(x,t) = \mid u_{ex}(x,t) - u_{app}(x,t) \mid,$$

and consequently, $||E||_\infty$ and $||E||_2$ error norms, are given by

$$||E||_\infty = \max\{E(x,t) : \ (x,t) \in \Omega \times [0,T]\},$$

$$||E||_2 = \left(\int \int_\Omega \left(u_{ex}(x,t) - u_{app}(x,t) \right)^2 dxdt \right)^{\frac{1}{2}}.$$

It is noted that, in this paper, the $||E||_2$ error norm is approximated by

$$||E||_2 \simeq \Big(\sum_{i=0}^{n} \frac{B-A}{2} w_i \sum_{j=0}^{m} \frac{T}{2}\omega_j (u_{ex}(\hat{\eta}_i, \hat{\theta}_j) - u_{app}(\hat{\eta}_i, \hat{\theta}_j))^2\Big)^{\frac{1}{2}},$$

where, $w_i = \frac{2}{n(n+1)} \frac{1}{(P_n(\eta_i))^2}$ and $\omega_j = \frac{2}{m(m+1)} \frac{1}{(P_m(\theta_j))^2}$ are LGL quadrature weights [10]. These norms are utilized to assess the precision of the method being suggested.

3.1. Example 1

Consider the nonlinear Phi-four equation (1) beside the boundary and initial conditions (2) and (3), where

$$\begin{aligned}
&\lambda_1 = \lambda_2 = 1, \ \lambda_3 = -1,\\
&\alpha = 2,\\
&s(x,t) = 0,\\
&A = 0, \ B = 1, \ T = 1,\\
&f_1(t) = \frac{3\lambda_2}{2\lambda_3}\Big(1 - \tanh^2\Big[\sqrt{\frac{\lambda_2}{4(4-\lambda_1)}}(A-2t)\Big]\Big),\\
&f_2(t) = \frac{3\lambda_2}{2\lambda_3}\Big(1 - \tanh^2\Big[\sqrt{\frac{\lambda_2}{4(4-\lambda_1)}}(B-2t)\Big]\Big),\\
&g_1(x) = \frac{3\lambda_2}{2\lambda_3}\Big(1 - \tanh^2\Big[\sqrt{\frac{\lambda_2}{4(4-\lambda_1)}}(x)\Big]\Big),\\
&g_2(x) = \frac{6\lambda_2}{\lambda_3}\Big(\sqrt{\frac{\lambda_2}{4(4-\lambda_1)}} \tanh\Big[\sqrt{\frac{\lambda_2}{4(4-\lambda_1)}}(x)\Big]\mathrm{sech}^2\Big[\sqrt{\frac{\lambda_2}{4(4-\lambda_1)}}(x)\Big]\Big).
\end{aligned}$$

This problem has been taken from [2] and has the exact solution

$$u(x,t) = \frac{3\lambda_2}{2\lambda_3}\Big(1 - \tanh^2\Big[\sqrt{\frac{\lambda_2}{4(4-\lambda_1)}}(x-2t)\Big]\Big).$$

Now, we address the problem utilizing the suggested approach. The approximated and exact solutions for $n = m = 14$ discretization points at both x and t axes, alongside the absolute error function, are shown in Fig. 1. In addition, the absolute error functions for $(n = 14, m = 7)$ and $(n = 7, m = 14)$ discretization points, are shown in Fig. 2. Also, $||E||_\infty$ and $||E||_2$ error norms obtained from the implementation of the proposed method for different values of n and m, are reported in Table 1, in comparison with those obtained in [2]. Moreover, in order to understand the rate of convergence, the log-linear graph of each obtained norms against the number of discretization points is plotted in Fig. 3. It is evident that, with a slight increase in discretization points, the error will decrease rapidly. These results confirm that, the accuracy as well as the rate of convergence of the proposed method is very suitable.

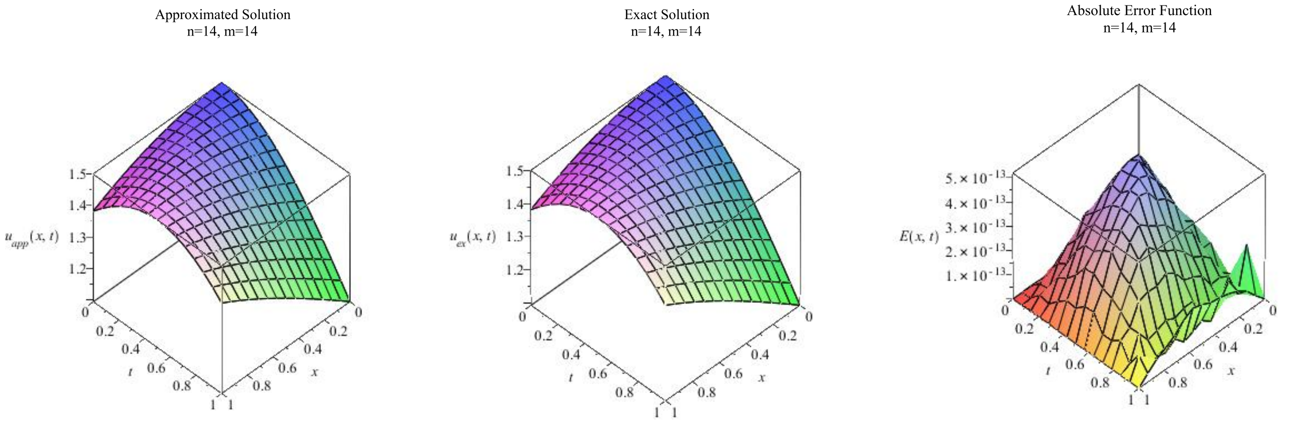

Figure 1. Solution of Example 1 for $n = m = 14$ discretization points.

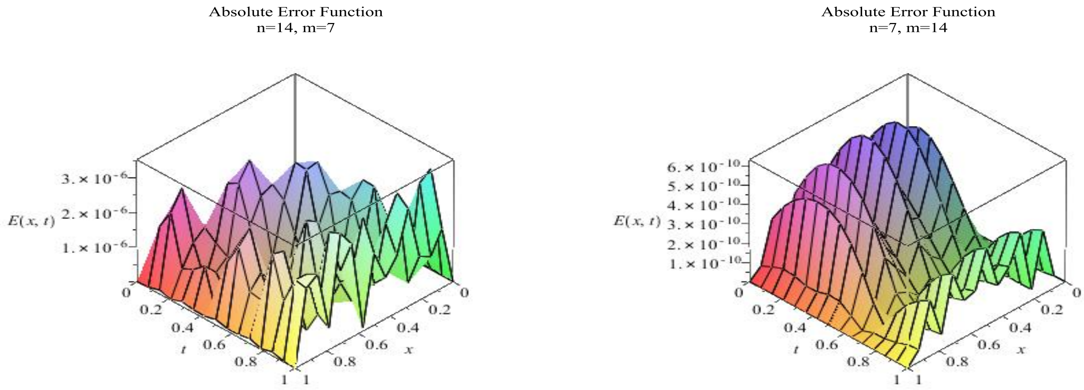

Figure 2. Absolute error functions in Example 1 for different values of n and m.

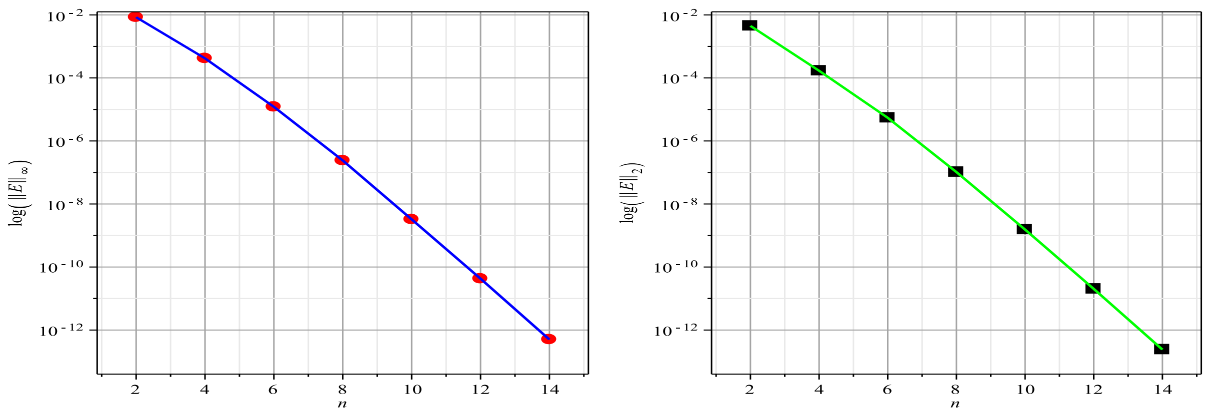

Figure 3. $||E||_\infty$ and $||E||_2$ versus the number of discretization points, in Example 1 (The log-linear graph).

Table 1. Error norms $||E||_\infty$ and $||E||_2$ for different values of n and m, in Example 1.

		Presented method		The best result in [2]	
n	m	$\|\|E\|\|_\infty$	$\|\|E\|\|_2$	$\|\|E\|\|_\infty$	$\|\|E\|\|_2$
2	2	8.45e-03	4.49e-03	Not reported	Not Reported
4	4	4.18e-04	1.69e-04	6.67e-05	Not Reported
6	6	1.21e-05	5.43e-06	Not reported	Not Reported
8	8	2.40e-07	1.03e-07	2.96e-10	Not Reported
10	10	3.25e-09	1.54e-09	Not reported	Not Reported
12	12	4.26e-11	2.02e-11	2.84e-10	Not Reported
14	14	5.00e-13	2.40e-13	Not reported	Not Reported

3.2. Example 2

In the second example, we examine the nonlinear Phi-four equation (1) along with the boundary and initial conditions (2) and (3), where

$$\begin{aligned}
&\lambda_1 = 1,\ \lambda_2 = -1,\ \lambda_3 = -1,\\
&\alpha = 3,\\
&s(x,t) = (-2 + x^2)\cosh(x+t) + x^6\cosh^3(x+t) - 4x\sinh(x+t),\\
&A = -1,\ B = 1,\ T = 1,\\
&f_1(t) = A^2\cosh(A+t),\\
&f_2(t) = B^2\cosh(B+t),\\
&g_1(x) = x^2\cosh(x),\\
&g_2(x) = x^2\sinh(x).
\end{aligned}$$

This problem has been taken from [9, 6, 26] and has the exact solution, $u(x,t) = x^2 \cosh(x+t)$. Now, we address the problem utilizing the suggested approach. The approximated and exact solutions for $n = m = 13$ discretization points at both x and t axes, alongside the absolute error function, are shown in Fig. 4. $||E||_\infty$ and $||E||_2$ error norms obtained from the implementation of the proposed method for different values of n and m, are reported in Table 2. Also, in Table 3, the error norms at $t = 1$ obtained from the proposed method in comparison with three other methods are given. Moreover, in order to understand the rate of convergence, the log-linear graph of each obtained norms against the number of discretization points is plotted in Fig. 5. It is seen that, the proposed method's accuracy is suitable and convergence is obtained with few discretization points.

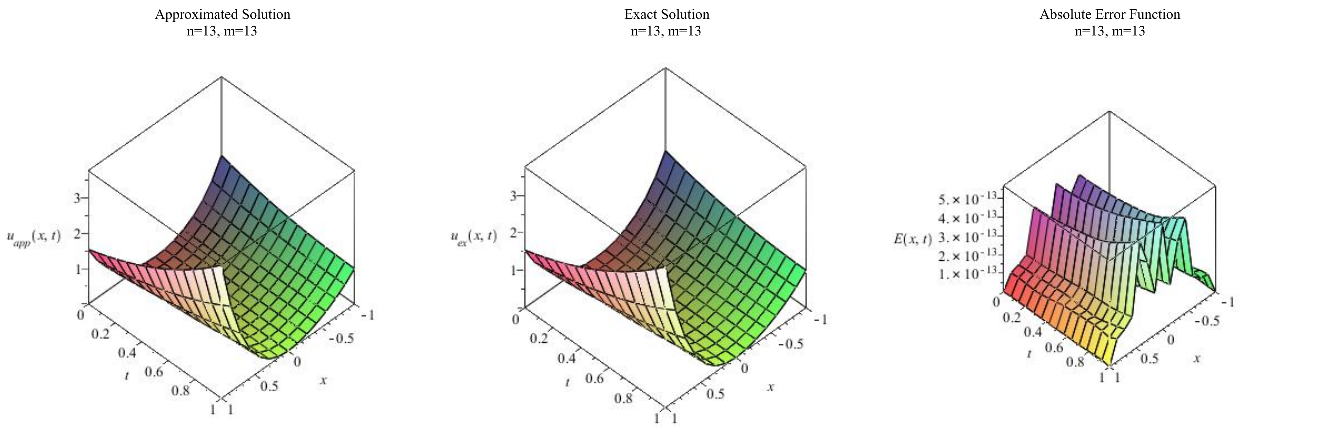

Figure 4. Solution of Example 2 for $n = m = 13$ discretization points.

Table 2. Error norms $||E||_\infty$ and $||E||_2$ for different values of n and m, in Example 2.

n	m	$\|\|E\|\|_\infty$	$\|\|E\|\|_2$
3	3	2.06e-01	2.12e-02
5	5	3.17e-03	1.62e-04
7	7	2.57e-05	6.15e-07
9	9	1.13e-07	1.45e-09
11	11	2.27e-10	2.65e-12
13	13	6.02e-13	1.83e-14

Table 3. Comparison between error norms $||E||_\infty$ and $||E||_2$ obtained from the proposed method at $t = 1$ and other methods, in Example 2.

$t = 1$	Result in [6]	Result in [26]	Result in [9]	The proposed method
$\|\|E\|\|_\infty$	5.07e-05	3.57e-06	1.03e-09	6.02e-13
$\|\|E\|\|_2$	2.95e-04	2.60e-05	3.71e-09	2.24e-14

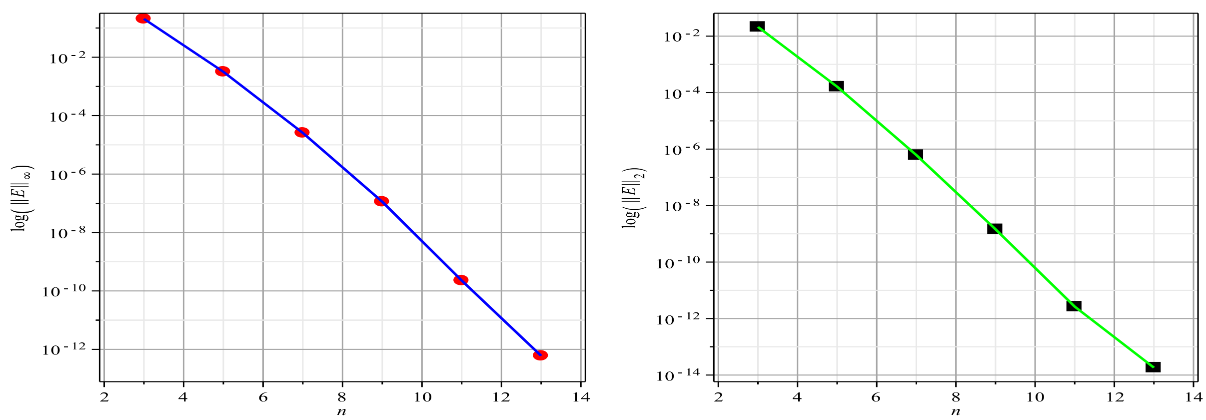

Figure 5. $||E||_\infty$ and $||E||_2$ versus the number of discretization points, in Example 2 (The log-linear graph).

4. Conclusion

In this study, an accurate and efficient pseudospectral method was applied for the numerical solution of nonlinear Phi-four equations that appear in particle physics and quantum mechanics. The main advantage of the proposed method is, its ability to achieve highly precise results with a small number of discretization points, and it demonstrates a high experimentally reported convergence rate. We believe that, the suggested approach can be expanded to solve various partial differential equations.

References

1. G. Arora and G. S. Bhatia. A meshfree radial basis function method for nonlinear Phi-four equation. *International Journal of Nonlinear Analysis and Applications*, 13(1):2043–2052, 2022.
2. A. H. Bhrawy, L. M. Assas, and M. A. Alghamdi. An efficient spectral collocation algorithm for nonlinear Phi-four equations. *Boundary value problems*, 2013:1–16, 2013.
3. C. Canuto, M. Hussaini, A. Quarteroni, and T. Zang. *Spectral Methods in Fluid Dynamics*. Springer Series in Computational Physics. Springer, Berlin, 1991.
4. A. Chowdhury and A. Biswas. Singular solitons and numerical analysis of ϕ–four equation. *Mathematical Sciences*, 6:1–8, 2012.
5. M. Dehghan and M. Shamsi. Numerical solution of two-dimensional parabolic equation subject to nonstandard boundary specifications using the pseudospectral Legendre method. *Numerical Methods for Partial Differential Equations*, 22(6):1255 1266, 2006.
6. M. Dehghan and A. Shokri. Numerical solution of the nonlinear Klein-Gordon equation using radial basis functions. *Journal of computational and Applied Mathematics*, 230(2):400–410, 2009.
7. M. Delkhosh and K. Parand. Generalized pseudospectral method: Theory and applications. *Journal of Computational Science*, 34:11–32, 2019.
8. S. T. Demiray and H. Bulut. Analytical solutions of Phi-four equation. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 7(3):275–280, 2017.
9. A. H. H. Encinas, J. Martín-Vaquero, A. Queiruga-Dios, and V. Gayoso-Martínez. Efficient high-order finite difference methods for nonlinear Klein-Gordon equations. I: Variants of the Phi-four model and the form-I of the nonlinear Klein-Gordon equation. *Nonlinear Analysis: Modelling and Control*, 20(2):274–290, 2015.
10. B. Fornberg. *A Practical Guide to Pseudospectral Methods*. Cambridge University Press, Cambridge, 1998.
11. B. Fornberg and D. M. Sloan. A review of pseudospectral methods for solving partial differential equations. *Acta numerica*, 3:203–267, 1994.
12. M. U. Haq, S. Haq, I. Ali, and M. J. Ebadi. Approximate solution of Phi-four and Allen-Cahn equations using non-polynomial spline technique. *Mathematics*, 12(6):798, 2024.
13. H. Heidary, M. Mehrpouya, A. Ghalee, and A. Oskouei. A design for a yoke to detect a notch on edge by using magnetic flux leakage method. *European Physical Journal Plus*, 136(4), 2021.
14. H. Karampour Beiranvand and M. A. Mehrpouya. An accurate h-pseudospectral method for numerical solution of the Bratu-type equations. *Computational Mathematics and Computer Modeling with Applications (CMCMA)*, 3(1):1–12, 2024.
15. M. M. Khater, A. Mousa, M. El-Shorbagy, and R. A. Attia. Analytical and semi-analytical solutions for Phi-four equation through three recent schemes. *Results in Physics*, 22:103954, 2021.
16. S. D. Kim, H.-C. Lee, and B. C. Shin. Pseudospectral least-squares method for the second-order elliptic boundary value problem. *SIAM Journal on Numerical Analysis*, 41(4):1370 1387, 2003.
17. D. Kondaxakis and S. Tsangaris. Pseudospectral solution of linear evolution equations of second order in space and time on unstructured quadrilateral subdomain topologies. *Journal of Computational Physics*, 202(2):533 576, 2005.
18. Z. Li, F. Zhang, and J. Zhou. Partial Newton-correction method for multiple fixed points of semi-linear differential operators by Legendre-Gauss-Lobatto pseudospectral method. *Journal of Scientific Computing*, 97(2), 2023.
19. H. Liu, S. L, H. Chen, and W. Chen. Gauss-Lobatto-Legendre-Birkhoff pseudospectral scheme for the time fractional reaction-diffusion equation with neumann boundary conditions. *International Journal of Computer Mathematics*, 96(2):362 378, 2019.
20. M. Mehrpouya. An efficient pseudospectral method for numerical solution of nonlinear singular initial and boundary value problems arising in astrophysics. *Mathematical Methods in the Applied Sciences*, 39(12):3204–3214, 2016.
21. M. Mehrpouya and H. Peng. A robust pseudospectral method for numerical solution of nonlinear optimal control problems. *International Journal of Computer Mathematics*, 98(6):1146–1165, 2021.
22. M. Mehrpouya and R. Salehi. A numerical scheme based on the collocation and optimization methods for accurate solution of sensitive boundary value problems. *European Physical Journal Plus*, 136(9), 2021.
23. M. Mehrpouya, M. Shamsi, and V. Azhmyakov. An efficient solution of hamiltonian boundary value problems by combined Gauss pseudospectral method with differential continuation approach. *Journal of the Franklin Institute*, 351(10):4765–4785, 2014.
24. M. A. Mehrpouya. A modified pseudospectral method for indirect solving a class of switching optimal control problems. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, 234(9):1531–1542, 2020.
25. M. Najafi. Using He's variational method to seek the traveling wave solution of Phi-four equation. *International Journal of Applied Mathematical Research*, 1(4):659–665, 2012.
26. J. Rashidinia, M. Ghasemi, and R. Jalilian. Numerical solution of the nonlinear Klein-Gordon equation. *Journal of Computational and Applied Mathematics*, 233(8):1866–1878, 2010.
27. M. Shahini and M. Mehrpouya. Transformed Legendre spectral method for solving infinite horizon optimal control problems. *IMA Journal of Mathematical Control and Information*, 35(2):341–356, 2018.
28. M. Shamsi and M. Dehghan. Recovering a time-dependent coefficient in a parabolic equation from overspecified boundary data using the pseudospectral Legendre method. *Numerical Methods for Partial Differential Equations*, 23(1):196 210, 2007.

29. M. Shamsi and M. Dehghan. Determination of a control function in three-dimensional parabolic equations by Legendre pseudospectral method. *Numerical Methods for Partial Differential Equations*, 28(1):74 93, 2012.
30. H. Triki and A.-M. Wazwaz. Envelope solitons for generalized forms of the Phi-four equation. *Journal of King Saud University-Science*, 25(2):129–133, 2013.
31. W. Zahra. Trigonometric B-spline collocation method for solving Phi-four and Allen-Cahn equations. *Mediterranean journal of mathematics*, 14:1–19, 2017.
32. W. Zahra, W. Ouf, and M. El-Azab. A robust uniform B-spline collocation method for solving the generalized Phi-four equation. *Applications and Applied Mathematics: An International Journal (AAM)*, 11(1):24, 2016.

