Research Article ISSN Online: 2783-4859 **Computational Mathematics and Computer Modeling with Applications**

Received 12 May 2023
Accepted 20 August 2024

DOI: 10.48308/CMCMA.3.1.13
AMS Subject Classification: 76M22; 34B15; 34B40; 65L60; 80A32

Exponential Gegenbauer collocation method for solving the MHD Falkner-Skan equation

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In this paper, we aim to introduce a weighted orthogonal system on the half-line based on the exponential Gegenbauer functions. We use these functions in collocation method to solve MHD Falkner-Skan equation, which arises in the study of laminar boundary layers exhibiting similarity on the semi-infinite domain. This method solves the problem on the semi-infinite domain without truncating it to a finite domain and transforming the domain of the problem to a finite domain. We make a comparison between the results of the proposed system with the numerical results to show that the present method has an acceptable accuracy. Copyright © 2024 Shahid Beheshti University.

Keywords: Exponential Gegenbauer; Collocation method; MHD Falkner-Skan equation; Semi-infinite domain; Nonlinear ODE.

1. Introduction

The Falkner-Skan equation was first introduced by Falkner and Skan [15], when studying the flow over a static wedge immersed in a viscous fluid. They applied a similarity transformation that can be used to reduce the partial differential boundary layer equations to a nonlinear third-order ordinary differential equation.

After that, interest and research in this equation and another boundary layer flows has increased [17, 25, 34, 38]. Liao [27] applied homotopy analysis method, which is independent of small or large physical parameters, to solve the Falkner-Skan equation and gave an explicit, totally analytical solution for this equation with the boundary conditions. Also, Asaithambi [3, 4] presented numerical methods based on finite-element, finite-differences methods. A differential transformation method, which obtains a series solution of the Falkner-Skan equation, is presented in [26].

In addition, developments in engineering have led to an increasing interest in magnetohydrodynamic (MHD) flows. The MHD viscous flows arise in many important engineering applications in devices such as power generators, the cooling of reactors, the design of heat exchanges, electrostatic filters, and MHD accelerators, among others [36]. The magnetic fields, one of the controlling forces, has stabilizing effects on the boundary layer flow [2]. A numerical method for the solution of the Falkner-Skan equation is presented in [5]. Yih [39] and Ishak et al. [24] transformed the partial differential boundary layer equations into the nonsimilar boundary layer equations and a system of ordinary differential equations respectively, then they used Keller box method to solve them. Hayat et al. [23] solved MHD boundary layer flow by modified decomposition method and padé approximation. Abbasbandy et al. [1, 2] applied a homotopy analysis method and Hankel-padé method respectively. Moreover, authors of [18] established the existence and uniqueness results for equations arising in MHD Falkner-Skan flow. Also, an approximate solution of this problem by Hermite functions pseudospectral method has been presented in [31]. Furthermore, some another numerical and analytical solutions also have been applied to different types of magnetohydrodynamic (MHD) flows problems [16, 29, 22, 32].

As we will describe in the next part of the study, the class of nonlinear third order ordinary differential equations arising in magnetohydrodynamic (MHD) Falkner-Skan flows is defined over the semi-infinite interval. In the context of spectral methods, a number of approaches for treating unbounded and semi-infinite domains have been proposed and investigated. Some common methods are using orthogonal polynomials which are determined over unbounded domains, The most common one is the use of polynomials that are orthogonal over unbounded domains [13, 28], mapping problem to a bounded domain [21] and domain truncation method [9]. Another effective direct approach for solving such problems is based on rational approximations. In this method, a set of new basis functions which are mutually orthogonal on unbounded domain is introduced and applied in a spectral scheme [12].

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Parand et al. [6, 7, 14, 30], also applied the spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on rational Tau and collocation methods.

The purpose of this paper is to develop the collocation method with new basis functions namely exponential Gegenbauer functions to solve MHD Falkner-Skan equation. Indeed, we use exponential transformation and have functions in semi-infinite domain to achieve high precision and exponential rate in convergence of solution.

The remainder of this paper is organized as follows: The mathematical description of MHD flow equation is presented in Section 2. Section 3 reviews the desirable properties of exponential Gegenbauer functions. In Section 4, we apply the exponential Gegenbauer collocation method in which we denote EGC to solve Falkner-Skan equation. Finally, Section 5 makes concluding remarks.

2. Mathematical description

Consider the steady laminar boundary layer flow of an electrically conducting viscous fluid in the presence of a magnetic field $B(x)$. The induced magnetic field is assumed to be small. This implies a small magnetic Reynolds number, so that magnetic field and Hall effect are neglected. Furthermore, the electric field as a result of polarization of charges is negligible. The governing equations within boundary layer approximation can be written as [2]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{du}{dx}+\nu\frac{\partial^2 v}{\partial y^2}-\frac{\sigma B^2}{\rho}(u-U), \tag{2}$$

$$\begin{aligned} &u=v=0 \ at \ y=0, \\ &u=U(x), \ \ when \ y\to\infty, \end{aligned} \tag{3}$$

where $U(x)=ax^m$ [33] and $B(x)=B_0x^{(m-1)/2}$ [11] and u and v are the velocity components, U is the inherent characteristic velocity, ν is a kinematic viscosity, σ is the electrical conductivity, ρ is the fluid density, B and B_0 are the magnetic field and externally imposed magnetic field in the y-direction respectively.

Defining

$$\tau=\sqrt{\frac{m+1}{2}}\sqrt{\frac{U}{\nu x}}y, \quad \psi=\sqrt{\frac{2}{m+1}}\sqrt{\nu x U}f(\tau), \tag{4}$$

$$u=Uf'(\tau), \qquad v=-\sqrt{\frac{m+1}{2}}\sqrt{\frac{\nu U}{x}}[f+\frac{m-1}{m+1}\tau f'], \tag{5}$$

the continuity equation is identically satisfied and Eq. (2) and boundary conditions Eq. (3) reduce to the following form

$$\frac{d^3 f}{d\tau^3}+f\frac{d^2 f}{d\tau^2}+\beta[1-(\frac{df}{d\tau})^2]-M^2(\frac{df}{d\tau}-1)=0, \tag{6}$$

with boundary conditions

$$f(0)=f'(0)=0, \quad f'(\infty)=1, \tag{7}$$

where $\beta=\frac{2m}{m+1}$ and $M^2=2\sigma B_0^2/\rho a(1+m)$. Now our focus is to find solution of Eq. (6) for wedge in the accelerated flow $(m>0,\beta>0)$ and decelerated flow $(m<0,\beta<0)$ with separation [2].

3. Exponential Gegenbauer interpolation

In this section, we present exponential Gegenbauer functions and express some of their basic properties of them. Then, we approximate a function using Gauss integration with exponential Gegenbauer-Gauss points.

3.1. Properties of exponential Gegenbauer functions

The Gegenbauer polynomials $G_n^\alpha(y)$ of order α and of degree n is defined as follow [37]:

$$G_n^\alpha(y) = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{\Gamma(n+\alpha-j)}{j!(n-2j)!\Gamma(\alpha)}(2y)^{n-2j}, \tag{8}$$

where n is an integer, α is real number greater than $-\frac{1}{2}$ and Γ is the Gamma function.

The Gegenbauer polynomials are orthogonal in the interval $[-1,1]$ with respect to the weight function $\rho(y) = (1-y^2)^{\alpha-\frac{1}{2}}$ where $\alpha > -\frac{1}{2}$. For a fixed α, they can be determined by the following recurrence formula [35]:

$$\begin{aligned} &G_0^\alpha(y) = 1, \qquad G_1^\alpha(y) = 2\alpha y, \\ &G_{n+1}^\alpha(y) = \frac{1}{n+1}\left[2y(n+\alpha)G_n^\alpha(y) - (n+2\alpha-1)G_{n-1}^\alpha(y)\right], \quad n \geq 1. \end{aligned} \tag{9}$$

For applying this polynomials in semi-infinite domain, the new basis functions denoted by $E_n^\alpha(x) = G_n^\alpha(y)$ is introduced where L is a constant parameter and $y = 1 - 2e^{-\frac{x}{L}}$, $y \in [-1,1]$. The constant parameter L sets the length scale of the mapping. Boyd [8] has offered some guidelines for optimizing the map parameter L for rational functions.

$E_n^\alpha(x)$ is the nth eigenfunction of the singular Sturm-Liouville problem:

$$4e^{-\frac{x}{L}}(1-e^{-\frac{x}{L}})E_n^{\alpha''}(x) - (2\alpha+1)(1-2e^{-\frac{x}{L}})E_n^{\alpha'}(x) + n(n+2\alpha)E_n^\alpha(x) = 0, \tag{10}$$

where the prime denotes differentiation with respect to x.

Exponential Gegenbauer functions satisfies in the following recurrence relation:

$$\begin{aligned} &E_0^\alpha(x) = 1, \qquad E_1^\alpha(x) = 2\alpha(1-2e^{-\frac{x}{L}}), \\ &E_{n+1}^\alpha(x) = \frac{1}{n+1}\left[2(1-2e^{-\frac{x}{L}})(n+\alpha)E_n^\alpha(x) - (n+2\alpha-1)E_{n-1}^\alpha(x)\right], \quad n \geq 1. \end{aligned} \tag{11}$$

3.2. Function approximation

We can determine $w(x) = \frac{2}{L}e^{-\frac{x}{L}}\left[4e^{-\frac{x}{L}}(1-e^{-\frac{x}{L}})\right]^{\alpha-\frac{1}{2}}$ as a non-negative, integrable, real-valued weight function for exponential Gegenbauer over the interval $I = [0,\infty)$.

Let us denote

$$\rho(y) = (1-y^2)^{\alpha-\frac{1}{2}}, \qquad y = 1-2e^{-\frac{x}{L}}, \tag{12}$$

hence, we have

$$\frac{dy}{dx} = \frac{2}{L}e^{-\frac{x}{L}}, \qquad \frac{dx}{dy} = -\frac{L}{(y-1)}, \qquad w(x)\frac{dx}{dy} = \rho(y). \tag{13}$$

Now, we define

$$L_w^2(I) = \{v : I \to \mathbb{R} \mid v \text{ is measurable and } \parallel v \parallel_w < \infty\}, \tag{14}$$

where

$$\parallel v \parallel_w = \left(\int_0^\infty \mid v(x) \mid^2 w(x)dx\right)^{\frac{1}{2}}, \tag{15}$$

is the norm induced by the scalar product

$$< u, v >_w = \int_0^\infty u(x)v(x)w(x)dx. \tag{16}$$

Thus, $\{E_n^\alpha(x)\}_{n\geq 0}$ denotes a system which is mutually orthogonal under Eq. (16), i.e.,

$$< E_n^\alpha, E_m^\alpha >_w = \frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}\delta_{nm}, \tag{17}$$

where δ_{nm} is the Kronecker delta function [35]. This system is complete in $L_w^2(I)$. For any function $u \in L_w^2(I)$, the following expansion holds

$$u(x) = \sum_{k=0}^{\infty} a_k E_k^\alpha(x), \tag{18}$$

with

$$a_k = \frac{< u, E_k^\alpha >_w}{\| E_k^\alpha \|_w^2}. \tag{19}$$

The a_k's are the expansion coefficients associated with the family $\{E_k^\alpha(x)\}$.

The differentiation formula for exponential Gegenbauer can be obtained as below:

$$E_n^{\alpha'}(x) = \frac{d}{dx} E_n^\alpha(x) = \frac{4\alpha}{L} e^{-\frac{x}{L}} E_{n-1}^{\alpha+1}(x). \tag{20}$$

So, we find that $E_n^{\alpha'}(x)$ also are mutually orthogonal in $L^2_{\widehat{w}}(I)$ with respect to the weight function $\widehat{w}(x) = \frac{L}{8\alpha^2} e^{-\frac{x}{L}} \left[4e^{-\frac{x}{L}}(1-e^{-\frac{x}{L}})\right]^{\alpha+\frac{1}{2}}$. Hence

$$< E_n^{\alpha'}, E_m^{\alpha'} >_{\widehat{w}} = \frac{\pi 2^{-(2\alpha+1)}\Gamma(2\alpha+n+1)}{(n-1)!(n+\alpha)[\Gamma(\alpha+1)]^2}\delta_{nm}. \tag{21}$$

3.3. Exponential Gegenbauer interpolation approximation

Authors of [10, 19] introduced Gauss integration. Here, we define exponential Gegenbauer-Gauss interpolation. Let

$$\mathfrak{E}_N^\alpha = \mathrm{span}\{E_0^\alpha, E_1^\alpha, ..., E_N^\alpha\}, \tag{22}$$

and y_j, $j = 0, 1, \dots, N$, be the $N+1$ roots of the polynomial $G_{N+1}^\alpha(y)$. These points are known as Gegenbauer-Gauss points. Their corresponding Christoffel numbers are [37]:

$$\frac{2^{2-2\alpha}\pi\Gamma(N+1+2\alpha)}{(N+1)![\Gamma(\alpha)]^2} \times \frac{1}{(1-y_j^2)[\frac{d}{dy}G_{N+1}^\alpha(y_j)]^2}. \tag{23}$$

We define

$$x_j = -L\ln\frac{1-y_j}{2} \quad j = 0, 1, \dots, N, \tag{24}$$

which are called exponential Gegenbauer-Gauss nodes. In fact, these points are zeros of the function $E_{N+1}^\alpha(x)$. Using Gauss integration, we have:

$$\begin{aligned} \int_0^\infty u(x)w(x)dx &= \int_{-1}^1 u\left(-L\ln\frac{1-y_j}{2}\right)\rho(y)dy \\ &= \sum_{j=0}^N u(x_j)w_j \quad \forall u \in \mathfrak{E}_{2N}^\alpha, \end{aligned} \tag{25}$$

where

$$w_j = \frac{2^{2-2\alpha}\pi\Gamma(N+1+2\alpha)}{(N+1)![\Gamma(\alpha)]^2} \times \frac{e^{-\frac{x_j}{L}}}{L^2(1-e^{-\frac{x_j}{L}})[\frac{d}{dx}E_{N+1}^\alpha(x_j)]^2}, \tag{26}$$

are the corresponding weights with the $N+1$ exponential Gegenbauer-Gauss nodes which can be obtained from Eqs. (23) and (24).

The interpolating function of a smooth function u on a semi-infinite interval is denoted by $P_N u$. It is an element of $\mathfrak{E}_N^\alpha$ and is defined as

$$P_N u(x) = \sum_{k=0}^N a_k E_k^\alpha(x). \tag{27}$$

$P_N u$ is the orthogonal projection of u upon $\mathfrak{E}_N^\alpha$ with respect to the inner product Eq. (16) and the norm Eq. (15) with $I = [0, \infty)$. Thus because of the orthogonality of the exponential Gegenbauer functions, we have [20]

$$< P_N u - u, E_i^\alpha >_w = 0 \quad \forall E_i^\alpha \in \mathfrak{E}_N^\alpha. \tag{28}$$

To apply a collocation method, we consider the residual function, $Res(x)$, when the expansion is substituted into the governing equation. The a_k's have to be selected so that the boundary conditions are satisfied, but the residual zero is made at as many (suitable chosen) spatial points as possible.

4. Solving Falkner-Skan equation

Now, we want to apply the exponential Gegenbauer collocation (EGC) method to solve the MHD Falkner-Skan equation introduced in Eq. (6) with the boundary condition in Eq. (7).

Falkner-Skan equation has a boundary condition like $f'(\infty) = 1$. So, we approximate $f(\tau)$ with $\tilde{P}_N f(\tau)$ operator:

$$\tilde{P}_N f(\tau) = \tau + \sum_{k=0}^{N} a_k E_k^{\alpha}(\tau), \tag{29}$$

then the residual function of MHD Falkner-Skan is:

$$\begin{aligned} Res(\tau) = &\frac{d^3}{d\tau^3}\tilde{P}_N f(\tau) + \tilde{P}_N f(\tau)\frac{d^2}{d\tau^2}\tilde{P}_N f(\tau) + \beta(1 - [\frac{d}{d\tau}\tilde{P}_N f(\tau)]^2) \\ &- M^2(\frac{d}{d\tau}\tilde{P}_N f(\tau) - 1). \end{aligned} \tag{30}$$

By solving the set of follow equations, we approximate $f(\tau)$ and succeed to obtain a_k's.

$$\begin{aligned} &Res(\tau_j) = 0, \quad j = 1, 2, ..., N-1, \\ &\tilde{P}_N f(0) = 0, \\ &\frac{d}{d\tau}\tilde{P}_N f(\tau)\Big|_{\tau=0} = 0 \end{aligned} \tag{31}$$

such as τ_j are exponential Gegenbauer-Gauss points.

In this paper, we solve Falkner-Skan equation for $m = 2$ with $M = 1, 2, 5, 10, 50, 100$ and $m = -3/5$ with $M = 3, 4, 5, 10, 15, 20, 50$ by equalizing α to 1 in EGC method and various L parameter which has been selected by using the incremental search method.

The physical quantities of interest represented by the value of $f''(0)$ is the skin friction coefficient. In Tables 1 and 2, we present values of $f''(0)$ that obtained by the method proposed in this paper for $N = 9$, $\alpha = 1$ and variant M for $m = 2$ and $m = -3/5$ respectively. We compare our results with the numerical value brought in [5] and results obtained by the homotopy analysis method (HAM) [2] in these Tables. It is shown that the value of $f''(0)$ increases with M. So, the presence of a magnetic field also increases the skin friction and the boundary layer thickness decreases while the skin friction increases.

Table 1. Comparison of $f''(0)$ for MHD Falkner-Skan equation when $m = 2$ between present method by $\alpha = 1$ and $N = 9$, HAM solution [2] and numerical results [5].

M	L	EGC	HAM	Numerical
1	1.48520	1.71946540	1.71947219	1.71946540
2	0.88870	2.43949833	2.43949870	2.43949833
5	0.52500	5.19095945	5.19095980	5.19095945
10	0.38090	10.09677545	10.09677575	10.09677545
50	0.10045	50.01944071	50.01944084	50.01944071
100	0.05006	100.00972170	100.00972177	100.00972170

Table 2. Comparison of $f''(0)$ for MHD Falkner-Skan equation when $m = -3/5$ between present method by $\alpha = 1$ and $N = 9$, HAM solution [2] and numerical results [5].

M	L	EGC	HAM	Numerical
3	1.62390	2.27338836	2.27338419	2.27338836
4	0.75600	3.48814857	3.48814572	3.48814857
5	0.59295	4.60075494	4.60075228	4.60075494
10	0.29693	9.80646420	9.80646300	9.80646420
15	0.19882	14.87167484	14.87167401	14.87167484
20	0.14933	19.90393701	19.90393626	19.90393701
50	0.09850	49.96165233	49.96165198	49.96165233

Tables 3 and 4 represent the coefficients of the exponential Gegenbauer functions obtained by the present method for some various values of M for $m = 2$ and $m = -3/5$, respectively. Taking these tables into account, one can see that a rapid convergence rate can be obtained by the implemented method even with small N.

Figures 1 and 2 show the velocity boundary layer of the wedge ($f'(\tau)$) with $m = 2$ and $m = -3/5$ for various M, respectively. According to these figures, The M parameter is directly related to the fluid velocity.

Table 3. Coefficients of the exponential Gegenbauer functions obtained by the present method for $m = 2$.

k	a_k $M = 1$	$M = 5$	$M = 100$
0	$-4.0038986944e-01$	$-1.4506084106e-01$	$-8.7116169937e-03$
1	$-1.0925473962e-01$	$-4.2677138776e-02$	$-1.6094397890e-03$
2	$5.1882881092e-02$	$1.7110940525e-02$	$1.0740181273e-03$
3	$-8.4348620153e-03$	$-2.2451211136e-03$	$-4.3029996993e-04$
4	$-1.4294689337e-03$	$-1.3528450761e-04$	$9.8059310914e-05$
5	$1.5709776978e-04$	$-1.3721578448e-05$	$-9.8602008486e-06$
6	$1.1223313109e-04$	$-2.0923890910e-06$	$4.1673710101e-09$
7	$3.0523943488e-05$	$-2.6795843395e-07$	$5.3159656285e-10$
8	$4.9529492128e-06$	$-1.4166451262e-08$	$6.3506911769e-11$
9	$3.7867583691e-07$	$1.8900016709e-09$	$9.6064567202e-12$

Table 4. Coefficients of the exponential Gegenbauer functions obtained by the present method for $m = -3/5$.

k	a_k $M = 3$	$M = 5$	$M = 50$
0	$-3.7242566782e-01$	$-1.6652118555e-01$	$-1.7390428859e-02$
1	$-9.1284596190e-02$	$-5.0163358752e-02$	$-3.2633391168e-03$
2	$4.7826599199e-02$	$1.9203779995e-02$	$2.1515696863e-03$
3	$-1.1026298019e-02$	$-2.2359776043e-03$	$-8.4459208440e-04$
4	$1.8751409647e-04$	$-1.1530609130e-04$	$1.8563716651e-04$
5	$-8.1833868063e-05$	$-4.1578620138e-05$	$-1.7192791185e-05$
6	$1.3969852283e-03$	$-6.3170419849e-06$	$-1.1449852684e-07$
7	$2.0380714255e-05$	$-1.6572245379e-06$	$-1.7144633772e-08$
8	$2.3755051305e-06$	$-3.5922426864e-07$	$-8.7164656535e-10$
9	$-6.6885828037e-07$	$-4.7089391297e-08$	$-2.8265562291e-10$

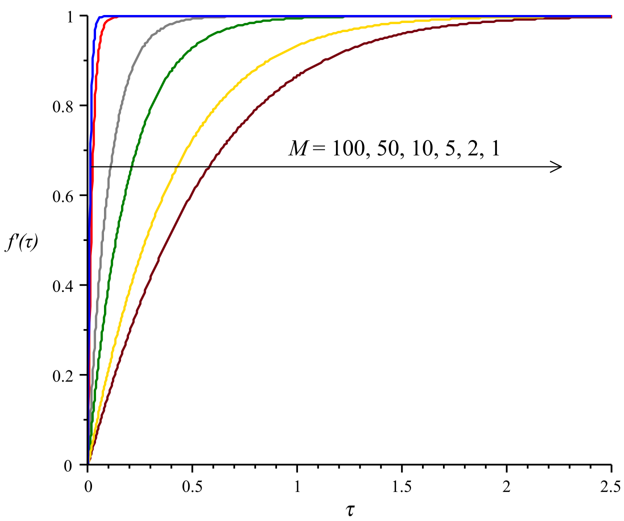

Figure 1. Graph of approximations of $f'(\tau)$, for Falkner-Skan equation with $m = 2$ and various M.

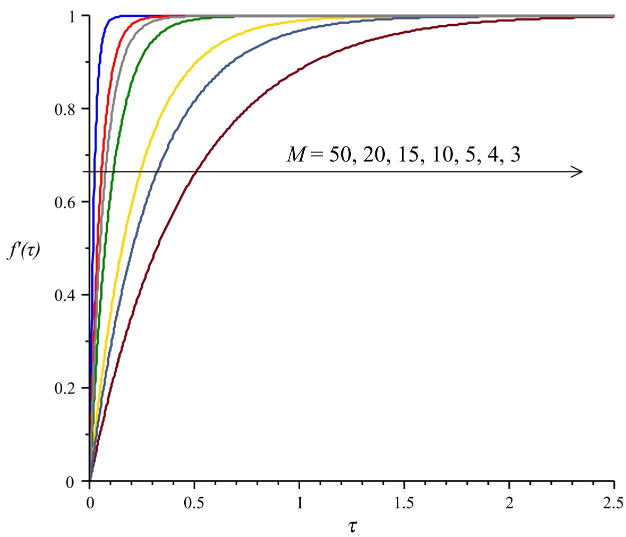

Figure 2. Graph of approximations of $f'(\tau)$, for Falkner-Skan equation with $m = -3/5$ and various M.

Finally, the logarithmic graph of absolute coefficients $|a_i|$ of the exponential Gegenbauer functions presented in Tables 3 and 4 are shown in Figures 3 and 4. These graphs illustrate the convergence rate of the method with a descending behavior and imply that accurate solutions can be obtained after relatively few iterations of the collocation method.

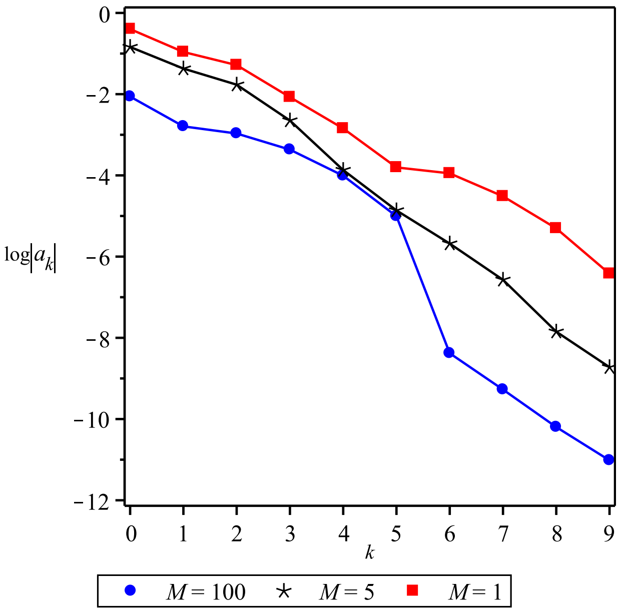

Figure 3. Logarithmic graph of absolute coefficients $|a_k|$ of the exponential Gegenbauer functions for $m = 2$.

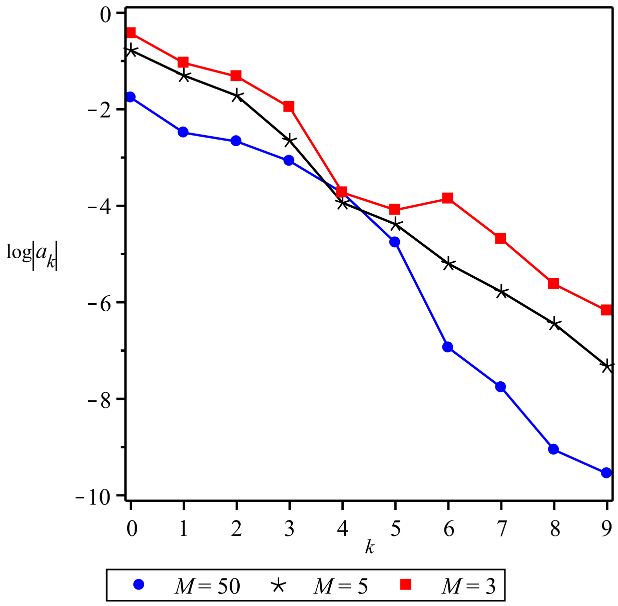

Figure 4. Logarithmic graph of absolute coefficients $|a_k|$ of the exponential Gegenbauer functions for $m = -3/5$.

5. Conclusions

In this paper, we introduced exponential Gegenbauer functions in $[0, \infty)$ interval and applied the collocation method based on these functions to solve the MHD Falkner-Skan equation. Collocation method is easy to implement and yields the desired accuracy. An important concern of the collocation approach is the choice of basis functions. The basis functions have three different properties: easy computation, rapid convergence and completeness, which means that any solution can be represented to arbitrarily high accuracy by taking the truncation N to be sufficiently large. We used exponential Gegenbauer functions as the basis function in this paper. Collocation method with these functions can solve the problems on the semi-infinite domain, such as boundary layer equation without truncating them to a finite domain, imposing the asymptotic condition and transforming domain of the problems. MHD Falkner-Skan equation arises in the study of laminar boundary layers. By selecting suitable values of characteristic velocity and an applied magnetic field, the resulting partial differential equations are reduced to a third-order nonlinear ordinary differential equation in the semi-infinite domain. The difficulty of applying these types of equations, due to the existence of its boundary condition in the infinity, is overcomed here. The validity of the method is based on the assumption that it converges by increasing the number of Gauss points. A comparison of the values of $f''(0)$ with numerical solution has shown that the new method is rapidly convergent.

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