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# An accurate h-pseudospectral method for numerical solution of the Bratu-type equations

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**In this study, an accurate method is developed for solving both initial and boundary value problems of the Bratu-type equations arising in various physical and chemical phenomena. In particular, we investigate the Bratu-Gelfand problem, which is of interest to many researchers because of the behavior of the solution. In the designed methodology, the problem is discretized using a h-pseudospectral method and therefore, solving the problem is reduced to solve a system of nonlinear equations. Numerical results of two examples are presented at the end and the comparison is made with the existing numerical or analytical solvers to show the efficiency and accuracy of the proposed method. Copyright © 2024 Shahid Beheshti University.**

**Keywords:** Bratu equations; Initial and boundary value problems; H-pseudospectral method; Legendre-Gauss-Radau points.

## 1. Introduction

In this paper, we investigate a method for obtaining the numerical solution of the one-dimensional Bratu-type equations, which have the form

$$u''(t) + \lambda e^{\mu u(t)} = 0, \quad 0 \le t \le 1, \tag{1}$$

where, $\lambda$ and $\mu$ are constants. The associated initial and boundary conditions are respectively given by

$$u(0) = \alpha, \;\; u'(0) = \beta, \tag{2}$$

$$\boldsymbol{\varphi}(u(0), u'(0), u(1), u'(1)) = \mathbf{0}, \tag{3}$$

where, $\boldsymbol{\varphi} : \mathbb{R}^4 \to \mathbb{R}^2$, is the boundary function of the problem. It is simple to show that, by using a substitution

$$u_1 = u, \tag{4}$$
$$u_2 = u', \tag{5}$$

we can reduce the order of the problem to one and therefore the following system of first-order differential equations is obtained

$$\begin{cases} u_1'(t) = u_2(t), \\ u_2'(t) = -\lambda e^{\mu u_1(t)}. \end{cases} \tag{6}$$

Consequently, the initial and boundary conditions (2) and (3) are changed to the conditions

$$u_1(0) = \alpha, \;\; u_2(0) = \beta \tag{7}$$
$$\boldsymbol{\varphi}(u_1(0), u_2(0), u_1(1), u_2(1)) = \mathbf{0}, \tag{8}$$

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respectively. Obviously, we can also express the system of differential equations (6) in the vector form

$$\mathbf{u}' = \mathbf{f}(t, \mathbf{u}), \tag{9}$$

where, $\mathbf{u}(t) = [u_1(t), u_2(t)]^T \in \mathbb{R}^2$, and

$$\mathbf{f}(t, \mathbf{u}) = \begin{bmatrix} u_2(t) \\ -\lambda e^{\mu u_1(t)} \end{bmatrix},$$

where, $\mathbf{f} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$.

The Bratu-type equations prove adequate for governing various aspects of science and engineering, especially steady state problems [65] and time independent processes under steady shear [17]. The Bratu problem is indeed a very good measure for testing the efficiency of various numerical techniques [44], especially for algorithm testing [67, 46]. The fact of the matter is that, the Bratu equation provides basis for shedding light on processes connected with electrospinning [61], mixed convection flows in vertical channels [51], electrically conducting solids [45], thermal explosion [25] thermal and chemical reaction [23], Chandrasekhar universe expansion model [15] and nanotechnology [1]. Due to the numerous applications, the Bratu-type equations have attracted considerable attentions. Therefore, many researchers have proposed different ways to obtain an approximate solution to this problem. In Table 1, we have tried to address almost all of the works done on solving various types of the Bratu problem.

The aim of this paper is to utilize one of pseudospectral methods which is based on using the Legendre-Gauss-Radau (LGR) points for accurate and efficient solution of both initial and boundary value problems of the Bratu-type equations. For this purpose, at first, the Eqs. (6) are discretized using the proposed method and next, the initial and boundary conditions (7) or (8) are considered for final solving of the Bratu problem.

It is noted that, the pseudospectral methods are classified among the main technologies for numerical solution of a wide variety of problems which classically were applied in fluid dynamics [14]. In this paper, we try to use one of these methods named as the h-pseudospectral method in which the main interval of the problem is divided into several subintervals and a fixed low degree polynomial is utilized in each subinterval. It is worthwhile to note that, the h-pseudospectral methods have a good history in solving the trajectory optimization problems (See for instance [31, 35, 47]). Therefore, we decided to use h-pseudospectral methods for the first time to solve the Bratu-type equations. An important advantage of using h-pseudospectral methods which are actually local methods, over the global pseudospectral methods is that, these methods lead to the production of smaller systems of algebraic equations than global methods which lead to very large systems of algebraic equations.

## 2. The proposed h-pseudospectral method

### *2.1. Preliminary considerations*

Implement the proposed h-pseudospectral method for the approximate solution of the Eq. (1), we first present some preliminaries about the pseudospectral methods. Indeed, for someone who is going to face such methods for the first time, two questions always stand out. The first question is that, how to approximate a function with these methods and the second question is how to explain the derivative of a function in the collocation points. In the following, we address these two questions.

Let $\{\zeta_i\}_{i=1}^{n}$ denote the LGR points for a certain positive integer $n$, in which $\zeta_1 = -1$, $\zeta_n < +1$ and a new point $\zeta_{n+1} = +1$ is also defined. So, we can easily approximate a function $\mathscr{F}$ on the interval $(-1, 1)$ by

$$\mathscr{F}(t) \simeq \sum_{i=1}^{n+1} \mathscr{F}(\zeta_i)\phi_i(t), \tag{10}$$

where, $\phi_i(t)$, $i = 1, \dots, n+1$, are the Lagrange polynomials which are based on $\{\zeta_i\}_{i=1}^{n}$ and an extra point $\zeta_{n+1} = +1$, that are established by

$$\phi_i(t) = \prod_{j=1, j\neq i}^{n+1} \frac{t - \zeta_j}{\zeta_i - \zeta_j}, \quad i = 1, \dots, n+1,$$

and have the Kronecker property

$$\phi_i(\zeta_j) = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases} \tag{11}$$

Moreover, for a function $\mathscr{H}$ on the arbitrary interval $(t_0, t_f)$, we have

$$\mathscr{H}(t) \simeq \sum_{i=1}^{n+1} \mathscr{H}(\hat{\zeta}_i)\hat{\phi}_i(t),$$

where, $\hat{\zeta}_i = ((t_f - t_0)\zeta_i + (t_f + t_0))/2$, $i = 1, \dots, n+1$ are the moved LGR points on $(t_0, t_f)$ and $\hat{\phi}_i(t) = \phi_i((2/(t_f - t_0))t - (t_f + t_0)/(t_f - t_0))$ are Lagrange polynomials related to $\hat{\zeta}_i$. Similarly, it can be concluded that, for a vector function $\mathbf{p}$ on the arbitrary interval $(t_0, t_f)$, we have

$$\mathbf{p}(t) \simeq \sum_{i=1}^{n+1} \mathbf{p}(\hat{\zeta}_i)\hat{\phi}_i(t). \tag{12}$$

  

**Table 1.** Review of the works done on solving various types of the Bratu problem.

| Approach | Authors/Year | Type of the equation | The method |
|---|---|---|---|
| Wavelet | Shahni and Singh [61]/2021 | 1D | Bernstein and Gegenbauer-wavelet collocation methods |
| | Swati et al. [67]/2021 | 1D | Modified Haar wavelet method |
| | Keshavarz et al. [37]/2018 | 1D | Taylor wavelets method |
| | Yang and Liao [75]/2017 | 1D | Wavelet homotopy analysis method |
| | Yang and Liao [76]/2017 | 2D | Wavelet homotopy analysis method |
| | Masood et al. [45]/2017 | 1D | Mexican hat wavelet method |
| | Liu et al. [42]/2013 | 2D | Modified wavelet Galerkin method |
| | Liu et al. [43]/2013 | 1D | Modified wavelet Galerkin method |
| | Hariharan and Pirabaharan [28]/2013 | 1D | Chebyshev wavelet method |
| | Venkatesh et al. [71]/2012 | 1D | Legendre wavelet method |
| | Venkatesh et al. [70]/2010 | 1D | Haar wavelet method |
| Spectral collocation (SC) | Abdelhakem and Youssri [4]/2021 | 1D | SC using Legendre polynomials |
| | Swaminathan et al. [66]/2021 | 1D | SC using Genocchi polynomials |
| | Singh [62]/2021 | 1D | Chebyshev SC |
| | Izadi and Srivastava [30]/2021 | 1D | SC using Bessel functions |
| | Doha et al. [21]/2013 | 1D | SC using Jacobi polynomials |
| | Boyd [11]/2011 | 1D | One-point pseudospectral collocation method |
| | Liu et al. [41]/2005 | 2D | SC using radial point interpolation |
| | Kapania [36]/1990 | 2D | Pseudospectral method |
| Optimal homotopy analysis (OHA) | Singh [63]/2020 | 1D | Green's function with OHA method |
| | Roul and Madduri [58]/2019 | 1D | OHA method |
| Finite difference (FD) | Aydinlik et al. [8]/2022 | 1D | Chebyshev FD method |
| | Iqbal and Zegeling [29]/2020 | 3D | Seven-point FD method |
| | Gharechahi et al. [24]/2019 | 1D | Compact FD method |
| | Hajipour et al. [26]/2018 | 1D, 2D and 3D | Fourth-order compact FD method |
| | Temimi and Ben-Romdhane [68]/2016 | 1D | Iterative FD method |
| | Al-Towaiq and Ala'yed [6]/2014 | 1D | FD cubic spline method |
| | Mohsen [49]/2014 | 1D and 2D | Nonstandard FD method |
| | Buckmire [12]/2005 | 1D | Mickens FD method |
| Hybrid block (HB) | Jator et al. [34]/2021 | 1D | HB 3rd-derivative Nyström method |
| | Rufai and Ramos [59]/2020 | 1D | One-step HB method |
| | Ramos and Rufai [55]/2020 | 1D | Optimized two-step HB method |
| | Jator and Manathunga [33]/2018 | 1D | Block Nyström method |
| Artificial neural network (ANN) | Raja et al. [54]/2016 | 1D | Feed-forward ANNs method |
| | Kumar and Yadav [39]/2015 | 1D | Multilayer perceptron ANN method |
| | Raja and Ahmad [53]/2014 | 1D | ANN method with various functions |
| | Raja [52]/2014 | 1D | Optimized ANN method |
| Galerkin & collocation (G & C) | Nabati and Nikmanesh [50]/2020 | 1D | Sinc-G/C methods |
| | Abd-Elhameed et al. [3]/2015 | 1D | G & C using Legendre polynomials |
| | Abd-Elhameed et al. [2]/2013 | 1D | G using Chebyshev polynomials |
| | Rashidinia et al. [56]/2013 | 1D | Sinc-G method |
| Optimization | Mehrpouya and Salehi [46]/2021 | 1D | Collocation & optimization methods |
| | Xu et al. [74]/2015 | 1D | Particle swarm shooting method |
| | Hamdi and Griewank [27]/2011 | 2D | FD & optimization methods |
| Variational iteration (VI) | Das et al. [18]/2016 | 1D | VI method |
| | Batiha [9]/2010 | 1D | VI method |
| Adomian decomposition (AD) | Singh et al. [64]/2015 | 1D | Green's function with AD method |
| | Wazwaz [72]/2005 | 1D | AD method |
| | Deeba et al. [19]/2001 | 1D | AD method |
| Spline Interpolation (SI) | Roul and Goura [57]/2022 | 1D | Sextic SI method |
| | Ala'yed et al. [7]/2019 | 1D | Quintic B-SI method |
| | Zarebnia and Sarvari [77]/2012 | 1D | Parametric SI method |
| | Caglar et al. [13]/2010 | 1D | B-SI method |
| | Jalilian [32]/2010 | 1D | Non-polynomial SI method |
| Others | Singh [44]/2022 | 1D | Homotopy perturbation method |
| | Tomar and Pandey [69]/2019 | 1D | Optimal Picard iteration method |
| | Deniz and Bildik [20]/2018 | 1D | Optimal perturbation method |
| | Wazwaz [73]/2016 | 1D | Successive differentiation method |
| | Sayevand et al. [60]/2015 | 1D | (G'/G)-expansion method |
| | Misirli and Gurefe [48]/2011 | 2D | Exp-function method |
| | Abbasbandy et al. [1]/2011 | 1D | Shooting method |
| | Aksoy and Pakdemirli [5]/2010 | 1D | Perturbation iteration algorithm |
| | Chang and Chang [16]/2008 | 1D | Differential transform method |
| | Li and Liao [40]/2005 | 1D | Homotopy analysis method |
| | Khuri [38]/2004 | 1D | Laplace transform method |
| | Boyd [10]/2003 | 1D | Chebyshev polynomial expansions method |

Now, with the knowledge of how to approximate a function by a pseudospectral method, we will discuss how to express the derivative of a function in the collocation points. In order to familiar with this aspect of the pseudospectral methods, by differentiating from (10), and then calculating it in $t = \zeta_j$, we have

$$\mathscr{F}'(\zeta_j) \simeq \sum_{i=1}^{n+1} d_{ij} \mathscr{F}(\zeta_i),$$

where

$$d_{ij} = \phi'_i(\zeta_j), \quad i, j = 1, \ldots, n+1, \tag{13}$$

is the $(i, j)$-th element of the $n+1$ by $n+1$ matrix $\mathbf{D}$, which is named as the differentiation matrix [22].

### 2.2. *Discretization of the Bratu-type equation*

We are ready to utilize the h-pseudospectral method for discretizing the problem (9). As it was said in the introduction, the main interval of the problem is supposed to be divided into several subintervals. So, if we set $t_0 = 0$ and $t_N = 1$, then the main interval $[0, 1]$ of the problem breaks into $N$ subintervals

$$[0, 1] = [t_0 = 0, t_1] \cup [t_1, t_2] \cup \ldots \cup [t_{N-1}, t_N = 1],$$

by calling an equally spaced grid

$$t_j = jh, \quad j = 0, 1, \ldots, N.$$

Consequently, the system of first-order differential equations (9) can be expressed by

$$\mathbf{u}'^{l}(t) = \mathbf{f}(t, \mathbf{u}^{l}(t)), \qquad t_l \leq t \leq t_{l+1}, \tag{14}$$

where, $l = 0, 1, \ldots, N-1$. Now, let $\hat{\zeta}_i^l$, $i = 1, \ldots, n$, $l = 0, 1, \ldots, N-1$, be the associated LGR points to the subinterval $[t_l, t_{l+1}]$, i.e. $\hat{\zeta}_i^l = ((t_{l+1} - t_l)\zeta_i + (t_{l+1} + t_l))/2$. The point $\hat{\zeta}_{n+1}^l = t_{l+1}$ is also added to the LGR points and then, a set of $n+1$ points, i.e., $\{t_l = \hat{\zeta}_1^l, \ldots, \hat{\zeta}_n^l, \hat{\zeta}_{n+1}^l = t_{l+1}\}$, associated with the subinterval $[t_l, t_{l+1}]$ is formed. This set of points is used for approximating each $\mathbf{u}^l(t)$, $l = 0, 1, \ldots, N-1$, as mentioned in Eq. (12) and so, we have

$$\mathbf{u}^{l}(t) \simeq \sum_{i=1}^{n+1} \mathbf{c}_i^l \hat{\phi}_i^l(t), \tag{15}$$

where, $\mathbf{c}_i^l = \mathbf{u}^l(\hat{\zeta}_i^l)$ are unknown parameters and

$$\hat{\phi}_i^l(t) = \phi_i^l\left(\frac{2}{t_{l+1} - t_l} t - \frac{t_{l+1} + t_l}{t_{l+1} - t_l}\right).$$

By using the Eq. (15), we can approximate $\mathbf{u}'^{l}$ by

$$\mathbf{u}'^{l}(t) \simeq \sum_{i=1}^{n+1} \mathbf{c}_i^l \hat{\phi}'^{l}_i(t). \tag{16}$$

Now, substituting approximations (15) and (16) into the Eq. (14), then collocating it in $\hat{\zeta}_k^l$, $k = 1, \ldots, n$, we get

$$\sum_{i=1}^{n+1} \mathbf{c}_i^l \hat{\phi}'^{l}_i(\hat{\zeta}_k^l) \simeq \mathbf{f}(\hat{\zeta}_k^l, \sum_{i=1}^{n+1} \mathbf{c}_i^l \hat{\phi}_i^l(\hat{\zeta}_k^l)), \ \ k = 1, \ldots, n, \ \ l = 0, 1, \ldots, N-1. \tag{17}$$

It is worthwhile to note that, although $\hat{\zeta}_{n+1}^l = t_{l+1}$ is utilized beside the moved LGR points for approximating $\mathbf{u}^l(t)$, but we do not use it for the collocation process. Now using the Eq. (13) and calling the Kronecker property in the Eq. (11), the Eq. (17) is converted to

$$\frac{2}{t_{l+1} - t_l} \sum_{i=1}^{n+1} \mathbf{c}_i^l d_{ik}^l - \mathbf{f}(\hat{\zeta}_k^l, \mathbf{c}_k^l) \simeq \mathbf{0}, \quad k = 1, \ldots, n, \ \ l = 0, 1, \ldots, N-1,$$

and finally the Eq. (14) is discretized. It is noted that, since the continuity condition of the solution of the Eq. (14) should be also considered, so, we have

$$\mathbf{u}^0(t_1) = \mathbf{u}^1(t_1),$$
$$\vdots$$
$$\mathbf{u}^{N-2}(t_{N-1}) = \mathbf{u}^{N-1}(t_{N-1}).$$

Consequently, using the Eqs. (15) and (11), we can obtain

$$\mathbf{c}_{n+1}^0 - \mathbf{c}_1^1 \simeq \mathbf{0},$$
$$\vdots$$
$$\mathbf{c}_{n+1}^{N-2} - \mathbf{c}_1^{N-1} \simeq \mathbf{0}.$$

  

### *2.3. Solution of the initial and boundary value Bratu-type problem*

Now, the time has come to finish solving the initial and boundary value Bratu-type problem by discretizing the initial and boundary conditions (7) and (8). Clearly, from the Eqs. (15) and (11), we have

$$\mathbf{u}^0(\hat{\zeta}_1^0 = 0) \simeq \sum_{i=1}^{n+1} \mathbf{c}_i^0 \hat{\phi}_i^0(\hat{\zeta}_1^0 = 0) = \mathbf{c}_1^0,$$

$$\mathbf{u}^{N-1}(\hat{\zeta}_{n+1}^{N-1} = 1) \simeq \sum_{i=1}^{n+1} \mathbf{c}_i^{N-1} \hat{\phi}_i^{N-1}(\hat{\zeta}_{n+1}^{N-1} = 1) = \mathbf{c}_{n+1}^{N-1}.$$

Finally, the Bratu-type problem (6) with the initial or boundary conditions (7) or (8) is respectively discretized to the following systems of nonlinear algebraic equations.

$$\mathbf{F}\begin{pmatrix} \mathbf{c}_1^0 \\ \vdots \\ \mathbf{c}_{n+1}^0 \\ \vdots \\ \mathbf{c}_1^{N-1} \\ \vdots \\ \mathbf{c}_{n+1}^{N-1} \end{pmatrix} = \begin{pmatrix} \frac{2}{t_1-t_0}\sum_{i=1}^{n+1} \mathbf{c}_i^0 d_{i1}^0 - \mathbf{f}(\hat{\zeta}_1^0, \mathbf{c}_1^0) \\ \vdots \\ \frac{2}{t_1-t_0}\sum_{i=1}^{n+1} \mathbf{c}_i^0 d_{in}^0 - \mathbf{f}(\hat{\zeta}_n^0, \mathbf{c}_n^0) \\ \vdots \\ \frac{2}{t_N-t_{N-1}}\sum_{i=1}^{n+1} \mathbf{c}_i^{N-1} d_{i1}^{N-1} - \mathbf{f}(\hat{\zeta}_1^{N-1}, \mathbf{c}_1^{N-1}) \\ \vdots \\ \frac{2}{t_N-t_{N-1}}\sum_{i=1}^{n+1} \mathbf{c}_i^{N-1} d_{in}^{N-1} - \mathbf{f}(\hat{\zeta}_n^{N-1}, \mathbf{c}_n^{N-1}) \\ \mathbf{c}_{n+1}^0 - \mathbf{c}_1^1 \\ \vdots \\ \mathbf{c}_{n+1}^{N-2} - \mathbf{c}_1^{N-1} \\ \hline \mathbf{c}_1^0 - [\alpha, \beta]^T \end{pmatrix} = \mathbf{0}, \tag{18}$$

$$\mathbf{F}\begin{pmatrix} \mathbf{c}_1^0 \\ \vdots \\ \mathbf{c}_{n+1}^0 \\ \vdots \\ \mathbf{c}_1^{N-1} \\ \vdots \\ \mathbf{c}_{n+1}^{N-1} \end{pmatrix} = \begin{pmatrix} \frac{2}{t_1-t_0}\sum_{i=1}^{n+1} \mathbf{c}_i^0 d_{i1}^0 - \mathbf{f}(\hat{\zeta}_1^0, \mathbf{c}_1^0) \\ \vdots \\ \frac{2}{t_1-t_0}\sum_{i=1}^{n+1} \mathbf{c}_i^0 d_{in}^0 - \mathbf{f}(\hat{\zeta}_n^0, \mathbf{c}_n^0) \\ \vdots \\ \frac{2}{t_N-t_{N-1}}\sum_{i=1}^{n+1} \mathbf{c}_i^{N-1} d_{i1}^{N-1} - \mathbf{f}(\hat{\zeta}_1^{N-1}, \mathbf{c}_1^{N-1}) \\ \vdots \\ \frac{2}{t_N-t_{N-1}}\sum_{i=1}^{n+1} \mathbf{c}_i^{N-1} d_{in}^{N-1} - \mathbf{f}(\hat{\zeta}_n^{N-1}, \mathbf{c}_n^{N-1}) \\ \mathbf{c}_{n+1}^0 - \mathbf{c}_1^1 \\ \vdots \\ \mathbf{c}_{n+1}^{N-2} - \mathbf{c}_1^{N-1} \\ \hline \boldsymbol{\varphi}(\mathbf{c}_1^0, \mathbf{c}_{n+1}^{N-1}) \end{pmatrix} = \mathbf{0}. \tag{19}$$

After solving the systems of nonlinear algebraic equations (18) or (19), the values of $\mathbf{c}_i^l$, $i = 1, \ldots, n+1$ and $l = 0, 1, \ldots, N-1$, are obtained and then the solution of the Bratu-type problem is completed.

## 3. Illustrative examples

In this section, two examples are provided to demonstrate the relevance and accuracy of the proposed method. For these examples, the system of nonlinear equations (18) or (19) is solved by the classical Newton's method

$$\mathbf{J}_{\mathrm{F}}(\mathbf{x}^{(k)})\mathbf{x}^{(k+1)} = \mathbf{J}_{\mathrm{F}}(\mathbf{x}^{(k)})\mathbf{x}^{(k)} - \mathbf{F}(\mathbf{x}^{(k)}), \quad k = 0, 1, \ldots.$$

where, $\mathbf{J}_{\mathrm{F}}(\mathbf{x}) = \frac{\partial \mathrm{F}(\mathrm{x})}{\partial \mathrm{x}}$ is the Jacobian matrix of $\mathbf{F}$ and during it's implementation, the elements of the Jacobian matrix are calculated by the central finite difference scheme as follows

$$\frac{\partial f_i(\mathbf{x})}{\partial x_j} \simeq \frac{f_i(\mathbf{x} + \mathbf{e}_j h) - f_i(\mathbf{x} - \mathbf{e}_j h)}{2h},$$

where, $\mathbf{e}_j$ is the unit vector whose the $j$-th element is one and all other elements are zeros and the step length value, $h$, is considered to be $10^{-04}$. In addition, all calculations are operated on a 2.53 GHz Core i5 PC Laptop with 4 GB of RAM running in Matlab R2016a.

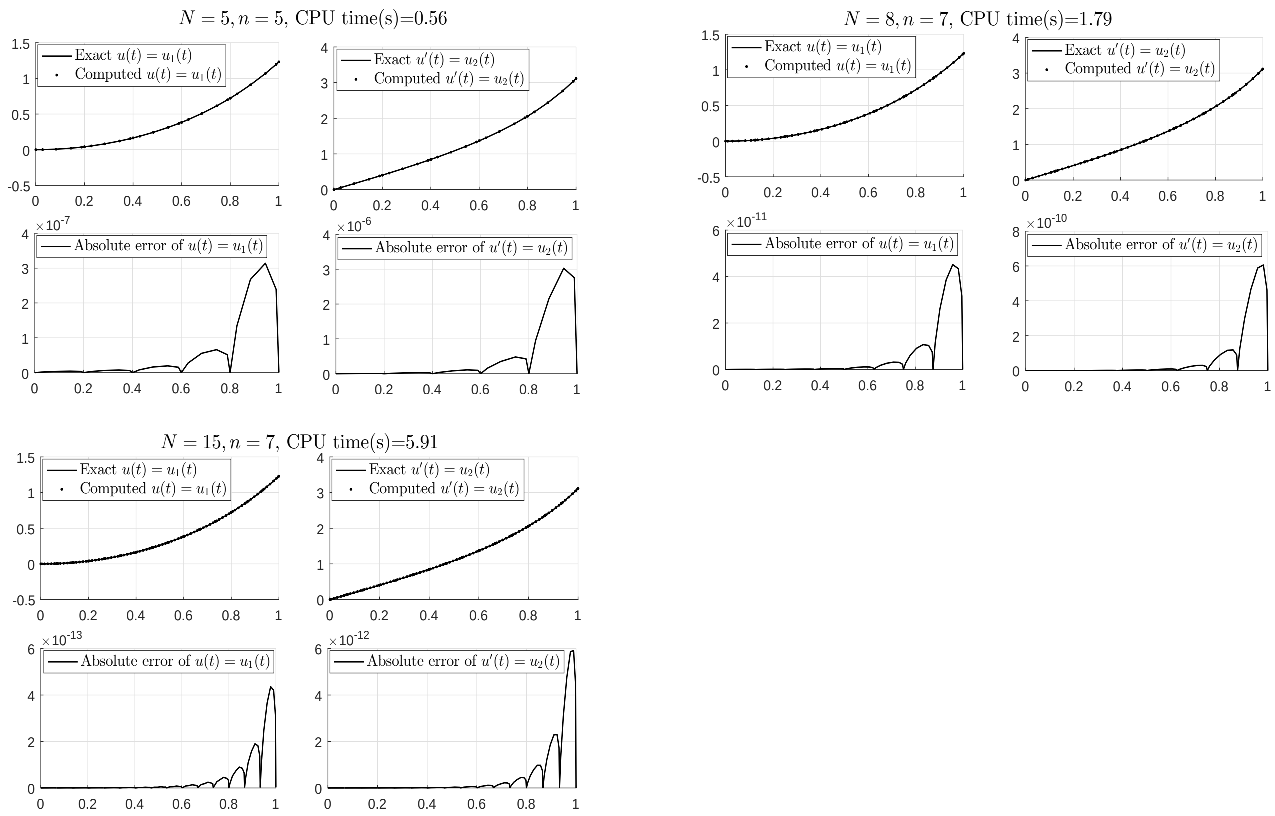


Figure 1. Solution of Example 1 for different values of $N$ and $n$.

### *3.1. Example 1: Initial value Bratu-type problem*

Consider the following initial value Bratu-type problem, which is taken from [71, 37]. The problem is presented by

$$\begin{aligned} &u''(t) - 2e^{u(t)} = 0, \quad 0 \le t \le 1, \\ &u(0) = u'(0) = 0. \end{aligned}$$

It is noted that, using the substitutions (4) and (5), the order of the problem is reduced to one, resulting in the following system of first-order initial value problems

$$\begin{cases} u_1'(t) = u_2(t), \\ u_2'(t) = 2e^{u_1(t)}, \\ u_1(0) = u_2(0) = 0. \end{cases}$$

This problem has the exact solution, $u_1^*(t) = -2\ln(\cos(t))$ and $u_2^*(t) = 2\tan(t)$. Now, let's solve the problem using the proposed method. In addition, the one vector, i.e. $[\mathbf{1}]_{2N(n+1)\times 1}$, is chosen as an initial guess for unknown parameters. The approximated solutions for different values of $N$ and $n$ alongside the exact solutions and the absolute error functions, are shown in Fig. 1. Also, the maximum absolute errors obtained from the implementation of the proposed method for different values of $N$ and $n$ with their CPU times, are reported in Table 2. In addition, the absolute errors between our results for $N = 15$ and $n = 7$ and for the function $u_1 = u$, are compared in Table 3 with the results in the other methods. This table confirms that, our proposed method has a very good numerical accuracy compared to the existing methods.

### *3.2. Example 2: Boundary value Bratu-type problem*

In the second example, we consider the classical Bratu problem, also known as the Bratu-Gelfand problem [26, 66]. The problem is presented by

$$\begin{aligned} &u''(t) + \lambda e^{u(t)} = 0, \quad 0 \le t \le 1, \\ &u(0) = u(1) = 0, \end{aligned}$$

and has the analytical solution, $u^*(t) = -2\ln\left[\frac{\cosh((t-\frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})}\right]$, where $\theta$ extracted from the equation $\theta = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$. This problem has attracted the attention of many researchers due to the behavior of it's solution. By looking at the Table 4, where the critical

**Table 2.** The maximum absolute errors $E(u_1)$ and $E(u_2)$ for different values of $N$ and $n$, in Example 1.

| $N$ | $n$ | E($u_1$) | E($u_2$) | CPU time(s) |
|---|---|---|---|---|
| 5 | 5 | 3.14e-07 | 3.03e-06 | 0.53 |
| 5 | 7 | 1.23e-09 | 1.61e-08 | 0.71 |
| 7 | 5 | 5.44e-08 | 5.39e-07 | 0.93 |
| 7 | 7 | 1.17e-10 | 1.57e-09 | 1.24 |
| 9 | 5 | 1.41e-08 | 1.42e-07 | 1.40 |
| 9 | 7 | 1.92e-11 | 2.59e-10 | 1.94 |
| 11 | 5 | 4.68e-09 | 4.75e-08 | 2.09 |
| 11 | 7 | 4.41e-12 | 5.97e-11 | 2.74 |
| 13 | 7 | 1.27e-12 | 1.73e-11 | 3.79 |
| 15 | 7 | 4.36e-13 | 5.91e-12 | 4.97 |

**Table 3.** Comparison of the absolute errors of $u_1 = u$ function in Example 1.

| $t$ | Present method | Taylor wavelets [37] | Best results of HPM [44] | AD [72] | Perturbation iteration algorithm [5] | GA [54] | GA-ASM [54] |
|---|---|---|---|---|---|---|---|
| 0.1 | 3.56e-16 | 7.80e-08 | 1.13e-16 | 2.98e-16 | 1.67e-13 | 5.85e-04 | 5.71e-08 |
| 0.2 | 2.08e-17 | 6.62e-08 | 0 | 8.57e-14 | 1.75e-10 | 6.19e-04 | 8.43e-08 |
| 0.3 | 7.36e-16 | 5.35e-08 | 5.55e-17 | 2.53e-11 | 1.04e-08 | 7.08e-04 | 7.18e-08 |
| 0.4 | 3.05e-16 | 1.77e-08 | 5.55e-17 | 1.46e-09 | 1.94e-07 | 8.03e-04 | 4.33e-08 |
| 0.5 | 1.89e-15 | 2.79e-08 | 1.67e-16 | 3.43e-08 | 1.93e-06 | 8.73e-04 | 1.30e-07 |
| 0.6 | 8.88e-16 | 5.29e-08 | 1.67e-16 | 4.61e-07 | 1.29e-05 | 9.42e-04 | 2.59e-07 |
| 0.7 | 1.27e-14 | 4.89e-08 | 3.15e-14 | 4.21e-06 | 6.63e-05 | 1.05e-03 | 5.89e-08 |
| 0.8 | 1.55e-15 | 3.80e-08 | 4.16e-12 | 2.92e-05 | 2.82e-04 | 1.21e-03 | 3.13e-07 |
| 0.9 | 1.13e-13 | 5.71e-08 | 3.16e-10 | 1.65e-04 | 1.05e-03 | 1.39e-03 | 1.43e-07 |

**Table 4.** The number of solutions of the Bratu-Gelfand problem and it's relationship with the value of $\lambda$.

| $\lambda$ | Number of solutions |
|---|---|
| $\lambda > \lambda_c$ | 0 |
| $\lambda = \lambda_c$ | 1 |
| $\lambda < \lambda_c$ | 2 |

value $\lambda_c$ satisfies the equation $1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh(\frac{\theta_c}{4})$ and is approximated by $\lambda_c = 3.513830719125161$ [26], the reason for this attention can be seen better. It is noted that, various numerical methods have been utilized for solving the Bratu-Gelfand problem and comparing the obtained results to it's analytical solution. Most of them converge to the first (lower) solution of this problem and unable to find the second (upper) solution of it, when $0 < \lambda < \lambda_c$. Another difficulty in the numerical solution of this problem is appeared, when $\lambda = \lambda_c$. In such a situation, there are two noteworthy points. The first is that, most of the existing methods presented to solve this equation have not investigated such a situation, and the second is that, the few existing methods have not been sufficiently accurate in solving this equation in this situation (See the Table 5).

Now, we intend to find both solutions of this problem, when $0 < \lambda < \lambda_c$ and also provide an accurate solution for this problem, when $\lambda = \lambda_c$. For this purpose, at first, this problem is solved for the small value $\lambda = 0.25$ and $\lambda = 1$. In Figs. 2 and 3, the approximated solutions for $u_1 = u$ are shown where the first (lower) solutions in both cases $\lambda = 0.25$ and $\lambda = 1$ are obtained with the one vector as an initial guess and the second (upper) solutions for the case $\lambda = 0.25$ and $\lambda = 1$ are obtained using the vectors with the elements 5 and 3, respectively, as an initial guess for unknown parameters. It is noted that, few existing methods in the literature, such as the method provided in [26], can find the second (upper) solution of this problem for the small values of $\lambda$. As we can see, the proposed method succeeds in finding both solutions of the Bratu-Gelfand problem with the appropriate accuracy, even for the small value of $\lambda$.

Moreover, the approximated solution of this problem in the case $\lambda = \lambda_c$ and accordingly $\theta_c = 4.798714561030936$, where the one vector is used as an initial guess for unknown parameters, is shown in Fig. 4. In addition, in Table 5, a comparison of the maximum absolute error is made between the proposed method and other numerical methods and softwares to find the first (lower) and unique solutions of the Bratu-Gelfand problem in the cases $\lambda = 1$ and $\lambda = \lambda_c$. The Table 5 confirms that, our proposed method has a very good numerical accuracy compared to the existing methods and softwares.

## 4. Conclusion

In this study, an accurate and precise h-pseudospectral method was applied for the numerical solution of both initial and boundary value problems of the Bratu-type equations that appear in a variety of important applications in science and engineering. The main advantages of the present work are that, a comprehensive survey was carried out on the methods and approaches for solving the Bratu-type equations and very good results are obtained along the low CPU time with the proposed method. It was also observed that, unlike many existing methods in the literature that only converge to the first (lower) solution of the Bratu

  

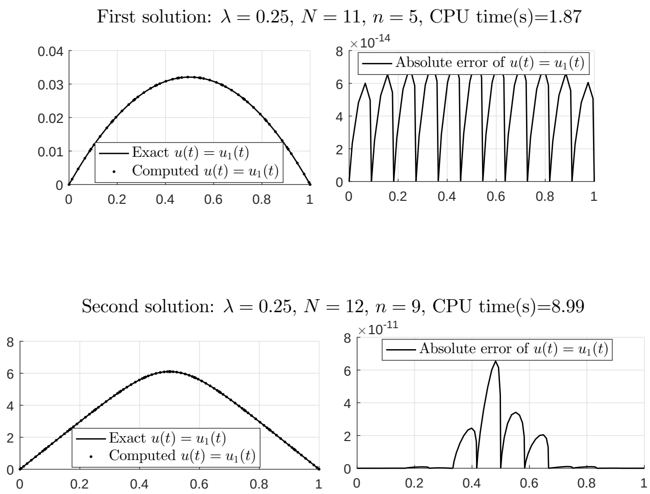


Figure 2. Solutions of Example 2 for $\lambda = 0.25$ and different of $N$ and $n$ via their CPU times(s).

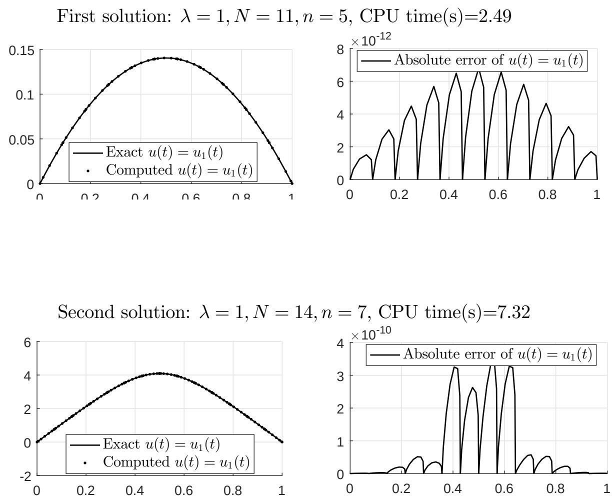


Figure 3. Solutions of Example 2 for $\lambda = 1$ and different values of $N$ and $n$ via their CPU times(s).

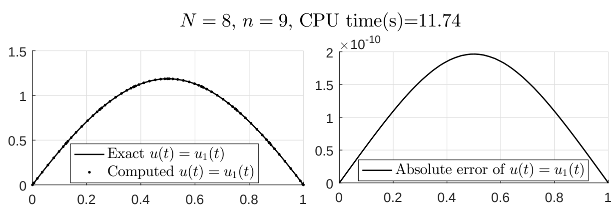


Figure 4. Solution of Example 2 for $\lambda = \lambda_c$ via it's CPU time(s).

problem, the proposed method is capable of finding the multiple solutions of the Bratu problem even for the small value of $\lambda$.

**Table 5.** Comparison of the maximum absolute error between the proposed method and other numerical methods and softwares to find the first (lower) and unique solutions of the Bratu-Gelfand problem in the cases $\lambda = 1$ and $\lambda = \lambda_c$, in Example 2.

| Authors/Year | $\lambda = 1$ | $\lambda = \lambda_c$ |
|---|---|---|
| Deeba et al. [19]/2001 | 2.69e-03 | Not reported |
| Kumar and Yadav [39]/2015 | 3.20e-03 | Not reported |
| Mohsen [49]/2014 | 1.00e-04 | 1.00e-01 |
| Khuri [38]/2004 | 1.35e-05 | Not reported |
| Das et al. [18]/2016 | 4.22e-05 | Not reported |
| Abbasbandy et al. [1]/2011 | 1.01e-06 | Not reported |
| Swaminathan et al. [66]/2021 | 8.01e-06 | Not reported |
| Caglar et al. [13]/2010 | 8.89e-06 | Not reported |
| Matlab function `bvp4c` | 2.47e-07 | 5.10e-02 |
| Roul and Madduri [58]/2019 | 2.60e-07 | Not reported |
| Xu et al. [74]/2015 | 1.24e-08 | 6.66e-04 |
| Buckmire [12]/2005 | 2.00e-08 | 4.00e-03 |
| Ala'yed et al. [7]/2019 | 9.90e-08 | Not reported |
| Mathematica function `NDSolve` | 5.71e-09 | 5.00e-08 |
| Temimi and Ben-Romdhane [68]/2016 | 5.71e-10 | Not reported |
| Jalilian [32]/2010 | 5.77e-10 | Not reported |
| Zarebnia and Sarvari [77]/2012 | 5.87e-10 | Not reported |
| Liu et al. [43]/2013 | 4.71e-11 | Not reported |
| Singh [44]/2022 | 1.25e-12 | Not reported |
| Ramos and Rufai [55]/2020 | 3.64e-12 | Not reported |
| Rashidinia et al. [56]/2013 | 6.50e-12 | Not reported |
| Keshavarz et al. [37]/2018 | 7.76e-12 | Not reported |
| Abdelhakem and Youssri [4]/2021 | 7.76e-12 | Not reported |
| Hajipour et al. [26]/2018 | 1.55e-14 | 4.11e-10 |
| Gharechahi et al. [24]/2019 | 4.69e-15 | Not reported |
| Nabati and Nikmanesh [50]/2020 | 5.33e-15 | 9.94e-06 |
| Swati et al. [67]/2021 | 5.60e-15 | Not reported |
| Jator and Manathunga [33]/2018 | 2.64e-16 | Not reported |
| Roul and Goura [57]/2022 | 5.51e-20 | Not reported |
| Tomar and Pandey [69]/2019 | 2.47e-27 | Not reported |
| The proposed method | 4.27e-16 ($N = 17$, $n = 7$) | 1.96e-10 ($N = 8$, $n = 9$) |

Furthermore, while many existing methods are unable to find an approximate solution of the problem with high accuracy in the case $\lambda = \lambda_c$, the proposed method succeeded in solving the problem accurately in this case. In addition, unlike many existing methods such as the global pseudospectral methods, the proposed method has very low sensitivity to the initial guess for solving the obtained system of algebraic equations.

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