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Developing Chimp Optimization Algorithm for Function Estimation Tasks

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This paper presents a novel approach for tackling the Lane-Emden equation, a significant nonlinear differential equation of paramount importance in the realms of physics and astrophysics. We employ the Chimp optimization algorithm in conjunction with Chebyshev polynomials to devise an innovative solution strategy. Inspired by the behavioral patterns of chimpanzees, the Chimp algorithm is harnessed to optimize the Chebyshev polynomial approximations, thereby transforming the Lane-Emden equation into an unconstrained optimization problem. Our method's effectiveness is demonstrated through a series of numerical experiments, showcasing its capability to precisely solve the Lane-Emden equation across various polytropic indices. Copyright © 2023 Shahid Beheshti University.

Keywords: Metaheuristic algorithms; Chimp optimization algorithm; Lane-Emden differential equations.

1. Introduction

Metaheuristic algorithms, which are a subset of artificial intelligence algorithms, are versatile tools for solving complex problems and finding applications in diverse fields such as optimization, networks, control, image processing, engineering design, finance, and machine learning. These algorithms are recognized for their powerful and efficient problem-solving capabilities, thanks to their ability to adapt to challenging and dynamic environments. They are invaluable assets for addressing optimization and complex problem scenarios in various domains [1]. Metaheuristic algorithms are powerful and versatile, drawing inspiration from natural and social phenomena. They can be categorized into four main types: physics-based, evolution-based, swarm-based, and human-based methods. The primary objective of these algorithms is to effectively explore and exploit the search space in order to find optimal solutions [33, 16]. By utilizing these metaheuristic algorithms, complex problems that traditional methods cannot handle efficiently can be effectively solved.

In nature, many organisms exhibit intelligent and cooperative behaviors that have inspired the development of optimization algorithms. For example, the concept of evolution has inspired genetic algorithms, while the social behavior of bees and ants has inspired bee colony optimization and ant colony optimization, respectively. Similarly, the Chimp Optimization Algorithm, a member of Swarm Intelligence-based Algorithms (SIAs) [5], has been inspired by the cooperative hunting behavior of chimpanzee populations in nature. Among SIAs like Particle Swarm Optimization (PSO) [14], Ant Colony Optimization (ACO) [8], and Artificial Bee Colony (ABC) [13], the Chimp Optimization Algorithm stands out as a novel approach that leverages the collective behavior of animals to balance exploration and exploitation and find near-optimal solutions [16]. Compared to other MOAs, SIAs have advantages such as scalability, memorylessness, and problem-agnosticism. Our task is to enhance the Chimp Optimization Algorithm [16] further and explore its potential in addressing challenging optimization problems. By observing and mimicking problem-solving strategies in nature, researchers have developed effective optimization algorithms that can be applied to various fields. Chimpanzees are known for their highly intelligent and cooperative behavior in nature, which includes communication, learning from each other, and collaborating to achieve common goals. These behaviors have inspired researchers to study and mimic the problem-solving strategies of chimpanzee communities in the development of optimization algorithms. The Chimp Optimization Algorithm (ChOA)[16] is one such algorithm that is inspired by the cooperative hunting behavior of chimpanzee populations in nature and is one member of Swarm Intelligence-based Algorithms (SIAs).

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By observing and mimicking the problem-solving strategies of chimpanzees, ChOA has been developed as an efficient and effective optimization algorithm that has shown promising results in solving various optimization problems in different fields. The main applications of ChOA belong to the domains of economics, image processing, engineering, neural network, power and energy, networks, etc [7]. It has also been used in research on health, environment, and public safety [7]. ChOA has been enhanced for continuous optimization domains and a novel Chimp Optimization Algorithm with sine and cosine has been proposed for solving complex tasks of different domains [12]. ChOA has use in various fields of application for example [3] enhanced chimp optimization algorithm (ChOA) using greedy search (GS) and opposition-based learning (OBL) to address real engineering problems efficiently, MOChOA [17] is a modified algorithm that improves multi-objective optimization by addressing the limitations of existing strategies, in [18] ULChOA is a variation of the Chimp Optimization Algorithm designed to address challenging constraint search spaces in real-world engineering problems and [10] NChOA is an improved version of ChOA that addresses multimodal problems by incorporating niching techniques and a local search approach.In another article [15] ChOA improves DCNN architecture for image classification, outperforming other models on benchmark datasets.A very importhan work is [4] that ChOA enhances COVID-19 detection with over 99.11% accuracy and clinically relevant Class Activation Map (CAM) results and in [29] An improved underwater image detection and recognition method using RBF neural networks and Chimp Optimization Algorithm (ChOA) achieves higher accuracy and outperforms previous models.Also researchers have explored the application of the Niching Penalized Chimp Algorithm for environmental economic dispatch optimization [34]. Additionally, a new binary meta-heuristic called the Binary Chimp Optimization Algorithm (BChOA) has been proposed as an effective approach for solving optimization problems [31].

Simulating the numerical solution of differential equations is a common approach used in metaheuristic algorithms. A differential equation [35] is an equation that involves derivatives of a dependent variable with respect to an independent variable. It can be classified based on its order and degree and can be ordinary or partial, linear or nonlinear. The solutions to differential equations are functions that satisfy the equation, with general solutions containing arbitrary constants and particular solutions containing specific values. Differential equations have extensive applications in mathematics, science, and engineering, serving to model exponential growth and decay, predict investment strategies, study medical phenomena, analyze electricity movement, understand economics, interpret wave motion, and tackle diverse engineering challenges. As a result, they are versatile and essential tools, empowering us to comprehend and forecast a wide range of natural and engineering phenomena.

To name a few applications of metaheuristic algorithms in the numerical solution of differential equations, various studies have utilized these powerful techniques. Sadollah [28] employed Particle Swarm Optimization with a Cuckoo Search algorithm to solve nonlinear differential equations. In another study, Harmony Search and Genetic Algorithm were used by Sadollah [27] to tackle longitudinal fins differential equations. Ebrahimzadeh [9] explored the application of three metaheuristic algorithms - multi-verse optimizer, moth-flame optimization, and a whale optimization algorithm - in solving fractional differential equations using a collocation-based approach. Additionally, the Cuckoo Optimization Algorithm has been developed for the numerical solution of Fredholm integral equations [6].

Furthermore, metaheuristic algorithms, particularly genetic algorithms, have shown promise in addressing the inverse heat conduction problem. This approach involves combining genetic algorithms with the least squares method to estimate temperature in the inverse parabolic problem. The genetic algorithm acts as a stochastic optimization method, effectively exploring the solution space for the best results. Notably, Pourgholi [25] and Mazraeh et al. [20] have discussed the use of genetic algorithms in conjunction with the sinc-galerkin method to solve inverse diffusion problems. Meanwhile, Hossein et al. [11] proposed a hybrid algorithm that combines particle swarm optimization (PSO) and genetic algorithms (GA) to enhance the accuracy and efficiency of solving inverse heat conduction problems. By leveraging the strengths of both PSO and GA, this metaheuristic approach aims to provide optimal solutions for complex inverse heat conduction problems.

In different domains, metaheuristic algorithms have found applications in optimization and prediction tasks. For instance, in [21] introduced the utilization of metaheuristic optimization algorithms, including Differential Evolution and Harris Hawks Optimization, in conjunction with a random forest model. This combined approach was employed in the field of aseismic materials to predict the energy transmission rate of an innovative concrete material with enhanced seismic resistance.

In this article, we aim to introduce a novel approach to solving differential equations using the ChOA, which has not been previously employed for this purpose. The proposed algorithm utilizes the Chebyshev polynomials as the basis function of function approximation. We will simulate different examples of well-known Lane-Emden nonlinear differential equations. Hence, the rest of the paper is organized as follows: In section two we call some preliminaries to our work, such as a detailed explanation of ChOA, the Lane-Emden equation, and the Chebyshev polynomials. In section 3, we explain our proposed methodology. Section 4, deals with the numerical results of the proposed approach, and in Section 5, some concluding remarks are noted.

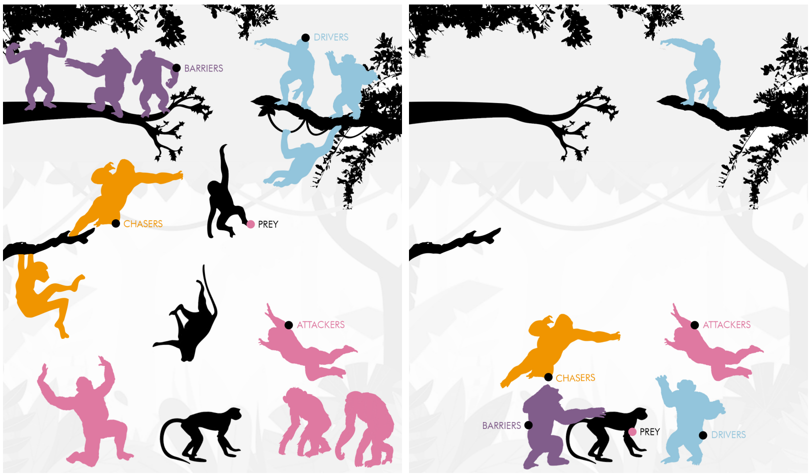
2. Preliminaries

2.1. Chimp Optimization Algorithm

Despite the fact that humans and chimpanzees exhibit several similarities, such as DNA, social behaviors, and cognitive abilities, there are notable differences between them. For example, humans have a more erect posture, a larger brain, and less body hair than chimpanzees. Moreover, humans possess advanced cognitive abilities, such as language, culture, and complex problem-solving skills, which are absent in chimpanzees. While scientific evidence suggests that chimpanzees can understand certain psychological processes, like vision, they do not perceive beliefs and other minds in the same way as humans [30].

Chimpanzees form complex social groups with a hierarchical structure, strong bonds, and various social behaviors like grooming and communication. Recent findings suggest they live in a **fission-fusion** society [16] and exhibit similarities to humans, including tool-making and cooperative hunting. However, conflicts can arise, leading to aggressive behaviors and territorial disputes. They also have impressive problem-solving abilities and cognitive skills, form family units, and interact with neighboring groups [22, 30]. In chimpanzee communities, **chimp leader** is the dominant individual who holds a central position within the social hierarchy. The leader exhibits dominant behavior, plays a role in decision-making, maintains social bonds, resolves conflicts, has priority in mating, and defends the group's territory. The chimp leader's characteristics ensure the survival and reproductive success of their community [32].

Chimpanzees are a species of great interest due to their remarkable interactions, communication, and intelligence. Their social behavior and problem-solving abilities make them a fascinating subject of study for scientists and researchers. According to [16], there are four distinct roles or types in chimpanzee societies as shown in 1:



(a) exploration (b) exploitation

Figure 1. The phases of hunting processes.

- **Drivers** are dominant chimpanzees that guide and direct their group's movements and actions. They play a crucial role in decision-making and organizing movements, such as leading the group to prey.
- **Chasers** are agile and fast chimpanzees that excel in pursuit and chasing behaviors. They are particularly efficient in pursuing specific targets, such as prey during hunting.
- **Barriers** are strong and assertive chimpanzees that protect and defend their community. They create barriers or obstacles to prevent intruders or threats from entering the community's territory, thus safeguarding the group and its territory.
- **Attackers** not only defend their group but also use their aggressive behavior to prognosticate the prey's breakout route. They can redirect the prey back towards the chasers or down into a lower canopy.

Corresponding ChOA algorithm, we have five parts in ChoA [16]:

1. **Driving and chasing the prey:** the process of driving and chasing prey by chimps is discussed. Mathematical equations (1) and (2) model this behavior, where $\mathbf{d}$ represents the distance between chimp and prey positions [16]:

$$\mathbf{d} = \left|\mathbf{c}.\mathbf{x}_{\text{prey}}(t) - \mathbf{m}.\mathbf{x}_{\text{chimp}}(t)\right|, \tag{1}$$
$$\mathbf{x}_{\text{chimp}}(t+1) = \mathbf{x}_{\text{prey}}(t) - \mathbf{a}.\mathbf{d}. \tag{2}$$

Here, t is the current iteration, $\mathbf{a}$, $\mathbf{m}$, and $\mathbf{c}$ are coefficient vectors, $\mathbf{x}_{\text{prey}}$ is prey position, and $\mathbf{x}_{\text{chimp}}$ is chimp position. The coefficients are calculated using equations (4), (5):

$$a = 2fr_1 - f, \tag{3}$$
$$\mathbf{c} = 2.\mathbf{r}_2, \tag{4}$$
$$\mathbf{m} = Chaotic, \tag{5}$$

Multiple independent Chimp groups with different strategies update f for local and global searches. This diversifies and balances search behavior. f is a key parameter in the optimization algorithm, regulating the balance between exploration and exploitation. It guides the algorithm's behavior in exploring solutions and is updated using varied strategies for local and global searches, thus improving optimization [16]. Independent groups enhance exploration, balance global-local search, and handle complex optimization. Chimps can change positions using random vectors. This process extends to n-dimensional spaces. Chimps also use chaotic strategies to attack prey, Chaotic refers to a state or behavior characterized by chaos, which is a complex and unpredictable pattern of behavior that appears random but is governed by underlying deterministic processes. We detailed in The fifth section sections.

2. **Attacking method (exploitation phase):** Chimps explore prey's location through driving, blocking, chasing, and encircling. Attacker chimps lead the hunting, supported by drivers, barriers, and chaser chimps. The best solutions inform the others. Equations (6)-(15) express their interactions [16]:

$$\mathbf{d}_{Attacker} = |\mathbf{c}_1\mathbf{x}_{Attac\,ker} - \mathbf{m}_1\mathbf{x} \tag{6}$$
$$\mathbf{d}_{Barrier} = |\mathbf{c}_2\mathbf{x}_{Barrier} - \mathbf{m}_2\mathbf{x}| \tag{7}$$
$$\mathbf{d}_{Chaser} = |\mathbf{c}_3\mathbf{x}_{Chaser} - \mathbf{m}_3\mathbf{x}| \tag{8}$$
$$\mathbf{d}_{Driver} = |\mathbf{c}_4\mathbf{x}_{Driver} - \mathbf{m}_4\mathbf{x}| \tag{9}$$

Updated positions is the process of adjusting the locations of chimp:

$$\mathbf{x}_1 = \mathbf{x}_{\text{Attacker}} - \mathbf{a}_1\left(\mathbf{d}_{\text{Attacker}}\right) \tag{10}$$
$$\mathbf{x}_2 = \mathbf{x}_{\text{Barrier}}\mathbf{a}_2(\mathbf{d}_{\text{Barrier}}) \tag{11}$$
$$\mathbf{x}_3 = \mathbf{x}_{Chaser} - \mathbf{a}_3(\mathbf{d}_{Chaser}) \tag{12}$$
$$\mathbf{x}_4 = \mathbf{x}_{Driver} - \mathbf{a}_4(\mathbf{d}_{Driver}) \tag{13}$$
$$\tag{14}$$

Overall updated position:

$$\mathbf{x}(t+1) = \frac{\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4}{4} \tag{15}$$

3. **Searching for prey (exploration):** In the final phase, the chimps initiate an attack when the prey halts its movement. To create a mathematical model of this attack, we adjust the f value, which in turn narrows the potential range for an effective, a becomes a variable with random attributes within the span of $[-2f, 2f]$, as f gradually decreases from 2.5 to 0 throughout the iterations. The use of random a values within $[-1, 1]$ strategically positions the chimp's next move between its present location and the prey's, ensuring a successful attack 1. ChOA employs specific methods to update chimp positions based on the positions of attackers, barriers, chasers, and drivers, driving the attack on the prey. However, to prevent getting stuck in localized solutions, additional techniques are required to promote exploration. While the driving, blocking, and chasing mechanisms provide a degree of exploration, ChOA would benefit from the incorporation of more methods to amplify this exploratory phase.
4. **Prey attacking (utilization):** During exploration, chimps emulate attacker, barrier, chaser, and driver chimps' paths to locate prey. They disperse following random values (> 1 or < -1), promoting global search by moving away from the prey. The c value in ChOA, as in equation (4), offers random weights $(0-2)$ to prey. It adjusts impact on distance in equation (5), enhancing stochastic behavior, and reducing local minima risks. c maintains randomness through iterations, is vital for end-phase exploration, and simulates obstacles hindering prey pursuit. It adapts prey's challenge according to the chimp's position.
5. **Social incentive (sexual motivation)):** As noted before, as chimps satisfy their nutritional and social needs, particularly through mating and grooming, their focus shifts away from hunting. ChOA use of chaotic maps to enhance ChOA's performance and simulate their behaviors. Six maps are utilized [16], These maps exhibit both deterministic and random behavior, with a common point of 0.7. To capture this combined behavior, a 50% probability governs the choice between normal position updates and chaotic models during chimp position adjustment as [16]:

$$\mathbf{x}_{chimp}(t+1) = \begin{cases} \mathbf{x}_{\text{prey}}(t) - \mathbf{a}.\mathbf{d} \text{ if } \mu < 0.5 \\ \textbf{Chaotic\_value} \text{ if } \mu \geq 0.5 \end{cases} \tag{16}$$

Update position normally if $\mu < 0.5$ and use chaotic value if $\mu \geq 0.5$ that μ is random in $[0, 1]$ and we calculate chaotic value by one of the six maps [16].

2.2. The Lane Emden Equation

The Lane-Emden equation [24] is a remarkable mathematical tool that has found a wide range of applications in physics and astrophysics. This nonlinear ordinary differential equation is particularly useful in modeling complex physical phenomena such as the structure of stars, the evolution of gas clouds, and the behavior of thermionic currents. One of the most interesting aspects of the Lane-Emden equation is its ability to describe the thermal history of a spherical cloud of gas, which is crucial in understanding the formation and evolution of celestial bodies. The equation is also used to model isothermal gas spheres and study their properties. In addition to its usefulness in astrophysics, the Lane-Emden equation has been studied extensively in theoretical physics, where it has been used to model a variety of physical systems. Its nonlinear nature and versatility make it a powerful tool for understanding complex physical phenomena.

The Lane-Emden equation describes the pressure and density distribution in a self-gravitating, spherically symmetric gas cloud, such as a star [24]. The pressure $P(r)$ at a distance r from the center of a spherical gas cloud is composed of two parts: regular gas pressure and an additional influence from radiation. We can express it as follows [24]:

$$P = \frac{1}{3}\xi T^4 + \frac{RT}{\nu}, \tag{17}$$

Here, ξ is a constant, T represents the temperature at distance r, R is the ideal gas constant, and ν is the specific volume. To analyze the system, we introduce dimensionless variables and describe the mass M of the fluid sphere based on its radius r. The following equations govern the variation of pressure and mass with respect to the radial distance [24]:

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}, \frac{dM(r)}{dr} = 4\pi\rho r^2, \tag{18}$$

Where G is the gravitational constant, and ρ denotes the density at a specific distance r from the central point of the star. We then relate the pressure and density changes with distance r by incorporating equation (18) into the hydrostatic equilibrium equation. This leads to the Lane-Emden equation, which is a modified form of the hydrostatic equilibrium equation [24]:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{19}$$

By introducing dimensionless parameters, we arrive at the dimensionless Lane-Emden equation as follows [24]:

$$\left[\frac{K(m+1)}{4\pi G}\lambda^{\frac{1}{m}-1}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dy}{dr}\right) = -y^m, \tag{20}$$

Where λ and K are constants, m is a parameter related to the density, and y is a dimensionless variable tied to ρ. The Lane-Emden equation is subject to certain boundary conditions. For positive values of x (where x is a dimensionless variable related to r), the equation with its derivatives expressed in terms of y, x, and their derivatives is:

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^m, \tag{21}$$

Together with the boundary conditions [24]:

$$y'' + \frac{2}{x}y' + y^m = 0, x > 0, \tag{22}$$

$$y(0) = 1, \quad y'(0) = 0. \tag{23}$$

To study the density distribution of a gas cloud for different values of m, we aim to solve equation (22) with the specified boundary conditions. The solutions obtained will allow us to better understand the structure of self-gravitating, spherically symmetric gas clouds, such as stars, under different conditions.

2.3. Chebyshev Polynomials

Chebyshev polynomials, named after Pafnuty Chebyshev, are a class of orthogonal polynomials with remarkable properties [19]. They arise in various mathematical and engineering contexts due to their advantageous properties, such as minimax approximation, orthogonality, and recurrence relations [26]. Here, we will provide an overview of Chebyshev polynomials and their key properties.

The n-th Chebyshev polynomial, denoted as $T_n(x)$, is defined on the interval $[-1, 1]$ as follows:

$$T_n(x) = \cos(n\cos^{-1}(x)), \tag{24}$$

where $n \in \mathbb{N}_0$ and $x \in [-1, 1]$. Chebyshev polynomials satisfy a recurrence relation given by:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{25}$$

with initial conditions $T_0(x) = 1$ and $T_1(x) = x$. One of the fundamental properties of Chebyshev polynomials is their orthogonality with respect to the weight function $w(x) = \frac{1}{\sqrt{1-x^2}}$. That is, for distinct integers m and n,

$$\int_{-1}^{1} T_m(x)T_n(x)w(x)dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\pi}{2}, & \text{if } m = n \neq 0, \\ \pi, & \text{if } m = n = 0. \end{cases} \tag{26}$$

Chebyshev polynomials have a unique property that makes them useful for polynomial approximation. The n-th Chebyshev polynomial minimizes the maximum absolute deviation (maximal error) from zero among all monic polynomials of degree n on the interval $[-1, 1]$. This can be achieved using the Chebyshev Interpolation Theorem.

Theorem 2.1 (Chebyshev Interpolation Theorem) *Let $f(x)$ be a continuous function on the interval $[-1, 1]$. Given a positive integer n, consider the Chebyshev nodes defined as*

$$x_k = \cos\left(\frac{(2k-1)\pi}{2n}\right), \quad \text{for } k = 1, 2, \ldots, n.$$

For any polynomial interpolation of degree at most n that interpolates the function f at these Chebyshev nodes, the resulting polynomial $P(x)$ minimizes the error in the maximum norm (sup-norm) sense, i.e.,

$$\max_{x\in[-1,1]} |f(x) - P(x)| = \min_{\substack{Q(x) \text{ is a polynomial} \\ \text{of degree at most } n}} \max_{x\in[-1,1]} |f(x) - Q(x)|.$$

In other words, choosing the Chebyshev nodes as interpolation points provides the optimal polynomial approximation of degree at most n in terms of minimizing the maximum absolute error.

Proof. This theorem is a classical result in numerical analysis. For a comprehensive proof and detailed discussions, we refer the interested reader to [26]. □

3. Methodology

To seamlessly integrate the ChOA into differential equation solving, we introduce a transformative adjustment to the conventional cost function. At the heart of this modification lies the chimp position vector, a foundational element in the ChOA's iterative optimization process. In a departure from its traditional role, the vector of length N is repurposed as weight coefficients for a Chebyshev polynomial of degree $n-1$. This strategic transformation empowers the optimization algorithm to iteratively modify these polynomial coefficients, thereby enabling efficient exploration of the solution space. Following this transformation, the estimated Chebyshev polynomial is seamlessly incorporated into the Lane-Emden Ordinary Differential Equation (ODE), giving rise to a residual function. This function acts as a quantifiable measure of the dissimilarity between the behavior of the estimated polynomial and the actual dynamics of the ODE. The overarching goal is to minimize this residual, achieved through the ChOA's iterative adjustments to the chimp position vector. Formally, we denote the approximation function as:

$$\hat{y}(x) = \sum_{i=0}^{N} w_i T_i(x).$$

Here w is the weight vector coming from the Chimp position and $T_i(x)$ is the i-th order Chebyshev polynomial. Substituting this approximation into the Lane-Emden ODE, the residual function takes the form:

$$Residual(x) = \hat{y}''(x_i) + \frac{2}{x_i}\hat{y}'(x_i) + \hat{y}(x_i)^m.$$

Integral to our methodology is the strategic selection of collocation points. These points should be drawn from the domain of the problem. Although any set of M distinct points can be chosen, the set of optimal training points can lead to better approximation. For example, another metaheuristic algorithm can be utilized. Here, we utilize the Chebyshev Interpolation Theorem and use Chebyshev nodes to guide the optimization process. By judiciously choosing these points, we enable the optimization algorithm to align the estimated Chebyshev polynomial with the Lane-Emden ODE at key intervals. This strategic alignment significantly enhances the precision and accuracy of the solution, effectively bridging the gap between the estimated and actual behaviors.

At the heart of the optimization process lies the computation of the cost associated with the Chimp position vector. This cost is calculated as the mean squared error between the Lane-Emden ODE and the estimated Chebyshev polynomial, evaluated at the selected collocation points. Formally, the cost can be expressed as:

$$Cost = \sum_{i=1}^{M} Residual(x_i)^2 + (\hat{y}(0) - 1)^2.$$

Here, M represents the number of collocation points. The ChOA utilizes this cost as a guiding metric for its iterative adjustments to the Chimp position vector. Through successive generations, the algorithm endeavors to minimize the cost function, thereby converging toward an optimal solution for the given differential equation.

Algorithm 1 Chimp Optimization Algorithm for Solving Differential Equation

1: Input Chimp position vector w
2: Form approximation function
3: Compute first derivative of approximation function
4: Compute second derivative of approximation function
5: Substitute $\hat{y}(x)$ into Lane-Emden ODE and obtain residual function
6: Generate collocation points $\{x_k\}_{k=1}^{M}$
7: Compute mean squared error (MSE) using collocation and residual functions
8: **return** MSE value

4. Numerical results

The Lane-Emden equation:

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, \quad x \geq 0, \tag{27}$$

under the constraint that $y(0) = 1$ and $y'(0) = 0$, for different values of m has analytical solutions. For $m = 0$, the exact solution is:

$$y(x) = 1 - \frac{x^2}{3!}. \tag{28}$$

For $m = 1$ we have:

$$y(x) = \frac{sin(x)}{x}, \tag{29}$$

and for $m = 5$, the nonlinear Lane-Emden equation has the fractional solution:

$$y(x) = (1 + \frac{x^2}{3})^{\frac{-1}{2}}. \tag{30}$$

For these examples, we report the absolute error of the prediction and exact behavior. The absolute error is computed as:

$$\text{Absolute Error} = |\text{Predicted Value} - \text{Exact Solution}|. \tag{31}$$

But since for $m = 2, 3, 4$, this equation does not have any known solution, we just report the approximated solution obtained through our computational method. These data, along with the optimization space of the ChOA, are visually presented in figures 2, 3, 4, 5, 6, and 7. These figures provide a comprehensive view of how the solutions evolve for different values of m. To facilitate a more detailed comparison and analysis, we have compiled the predicted values of these equations into Table 2. This table includes predicted values for $m = 2, 3$ and 4, allowing for a convenient side-by-side examination of the results. In all of these examples, we carefully selected the parameters of the ChOA as specified in Table 1. The selection of these parameters plays a crucial role in the optimization process and the accuracy of our predictions [2].

Expanding our analysis to non-integer values of m, such as $m = 1.5$, is of great importance in scientific and engineering applications. These non-integer values often arise in real-world problems where the Lane-Emden equation serves as a mathematical model. Solving for such values of m can shed light on the behavior of physical systems that exhibit fractional or intermediate order dynamics.

In our approach to tackling $m = 1.5$, we explore various numbers of basis functions to better understand the nuances of the solution. This exploration provides valuable insights into how the Lane-Emden equation behaves in regions of non-integer orders, and it also allows us to assess the accuracy and efficiency of our computational method in such scenarios.

Table 3 serves as a repository of our findings for $m = 1.5$. By presenting the results for different numbers of basis functions, we not only demonstrate the adaptability of our approach but also offer researchers and practitioners a valuable resource for future comparisons and benchmarking. In cases where benchmark solutions or known results are available, our table provides a basis for evaluating the accuracy and reliability of our method.

Parameter	Value
Number of Chimps	500
Upper bound	1
Lower bound	-1
Maximum number of Chimps	2200

Table 1. Parameter of ChOA algorithm

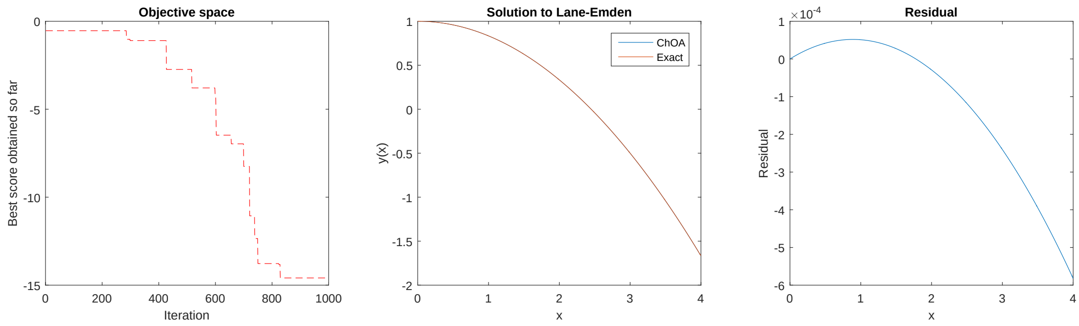


Figure 2. Results for Lane-Emden equation with $m = 0$. The value of the cost function in logarithmic scale (left) The exact and the approximate solution (Center) The absolute error (Right)

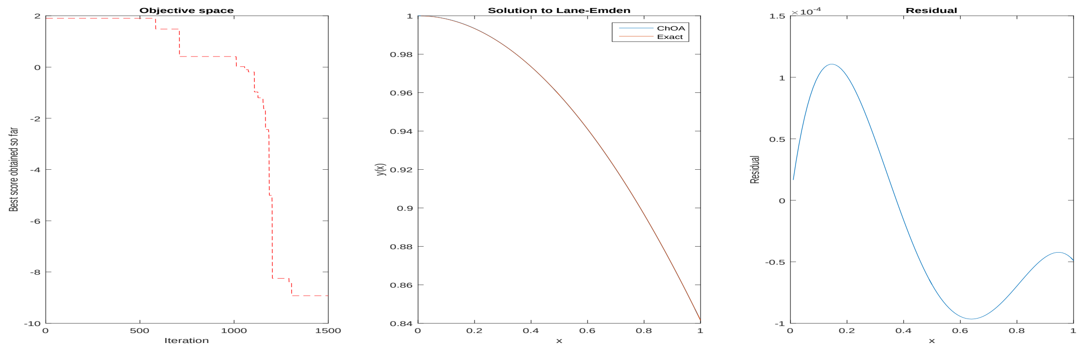


Figure 3. Results for Lane-Emden equation with $m = 1$. The value of the cost function in logarithmic scale (left) The exact and the approximate solution (Center) The absolute error (Right)

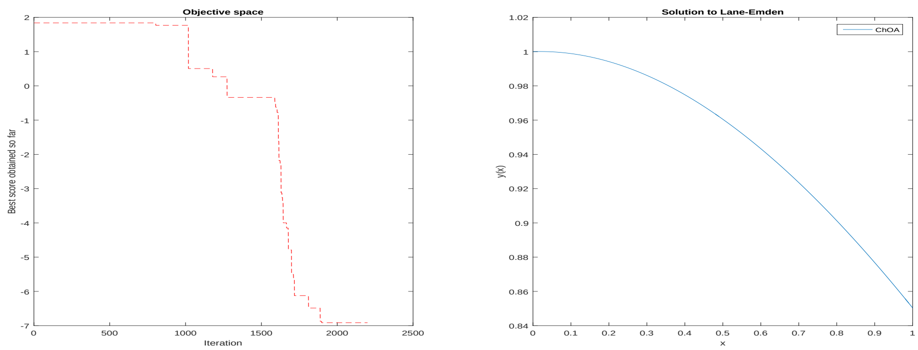


Figure 4. Results for Lane-Emden equation with $m = 2$. The value of the cost function in logarithmic scale (left) The approximate solution (Right)

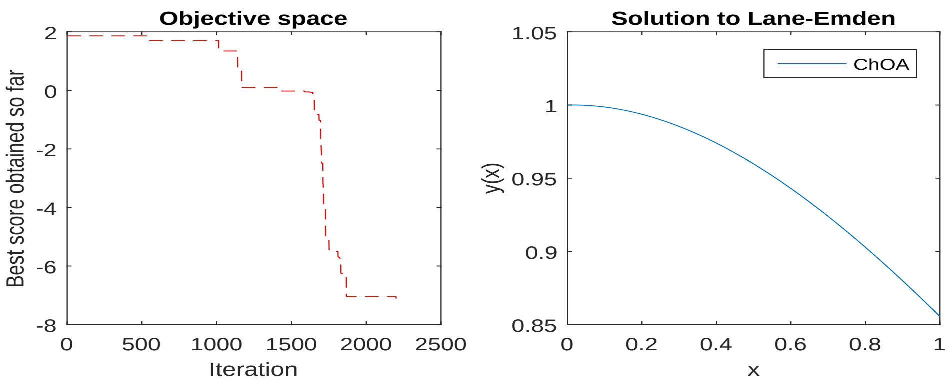


Figure 5. Results for Lane-Emden equation with $m = 3$. The value of the cost function in logarithmic scale (left) The approximate solution (Right)

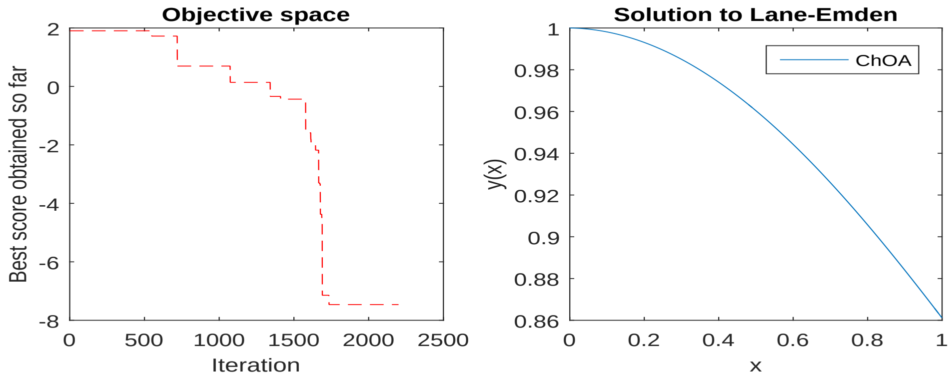


Figure 6. Results for Lane-Emden equation with $m = 4$. The value of the cost function in logarithmic scale (left) The approximate solution (Right)

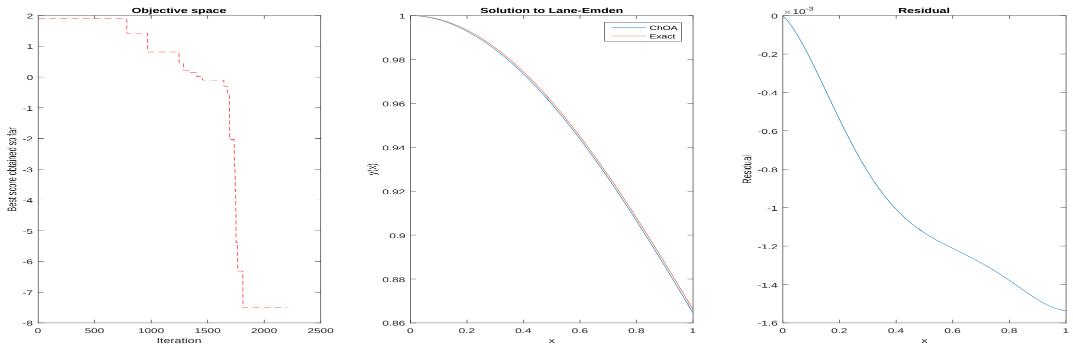


Figure 7. Results for Lane-Emden equation with $m = 5$. The value of the cost function in logarithmic scale (left) The exact and the approximate solution (Center) The absolute error (Right)

	$m = 2$		m=3		m=4	
x	[23]	Presented Method	[23]	Presented Method	[23]	Presented Method
0..01	0.99998	0.99991	0.99998	0.99995	0.99998	0.99975
0.02	0.99998	0.99993	0.99993	0.99997	0.99993	0.99987
0.05	0.99958	0.99954	0.99958	0.99943	0.99958	0.99946
0.1	0.99833	0.99829	0.99833	0.99824	0.99833	0.99814
0.5	0.95935	0.95939	0.95983	0.95973	0.95983	0.95943
1	0.84864	0.84851	0.85516	0.85556	0.85516	0.85546

Table 2. Comparison between numerical results obtained by the ChOA and neural networks for $m = 2, 3, 4$.

x	$N = 2$	$N = 3$	$N = 4$	$N = 5$
0.01	0.8535	0.9998	1.0000	0.9998
0.02	0.8519	0.9997	1.0000	0.9997
0.05	0.8472	0.9990	0.9997	0.9992
0.10	0.8392	0.9973	0.9985	0.9983
0.50	0.7755	0.9591	0.9584	0.9715
1.00	0.6959	0.8501	0.8442	0.8602

Table 3. The numerical values obtained from ChOA algorithm for the Lane-Emden equation with $m = 1.5$

5. Conclusions

In conclusion, our study introduces a pioneering computational framework for addressing the Lane-Emden equation, a pivotal nonlinear differential equation with wide-ranging applications in astrophysics and mathematical physics. By synergistically employing the Chimp optimization algorithm and Chebyshev polynomial expansions, we have demonstrated a robust methodology for approximating the solution to this equation. The formulation as an unconstrained optimization problem, guided by the exploration-exploitation dynamics of the Chimp algorithm, showcases the efficacy of our approach in navigating complex solution spaces. Notably, our method yields remarkably accurate solutions for both benchmark examples, in instances where an analytical solution is available and otherwise. These results underscore the viability and versatility of our proposed technique, shedding light on its potential for efficiently tackling nonlinear equations in theoretical physics. Our findings further underscore the efficacy of bio-inspired optimization algorithms in enhancing scientific computing methodologies.

Nevertheless, several limitations can pose challenges to the simulation process. One such constraint pertains to the necessity of defining bounds for each dimension, a task that can prove problematic. Employing narrow bounds may result in imprecise solutions, while adopting broad intervals can impede the optimization process. Additionally, the algorithm's time complexity escalates when addressing high-dimensional systems, warranting the consideration of strategies like the utilization of parallel algorithms, such as the multi-core parallelization algorithm outlined in [25], to mitigate this issue.

In the context of future research endeavors, we propose extending the application of this algorithm to solve a broader spectrum of problems, encompassing partial differential equations, integral equations, and fractional differential equations. Furthermore, we suggest exploring alternative basis functions, such as fractional and rational Jacobi functions, to assess their impact on the algorithm's convergence and accuracy.

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