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# On modified decomposition of interval matrices and its application

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**In this paper we present the LU decomposition of a generalized interval matrix A under a modified interval arithmetic. This modified interval arithmetic is defined on generalized intervals and possesses group properties with respect to the addition and multiplication operations. These properties cause that the two computed generalized interval matrices L and U from the LU decomposition satisfy A = LU, with equality in modified interval arithmetic instead of the weaker inclusion in the classical interval arithmetic. Some applications of the new technique for solving interval linear systems are given and effectiveness of the new approach is investigated along some numerical tests.** 

**Keywords:** Decomposition of matrices; Interval linear systems.

## 1. Introduction

There are numerous applications of matrices in various branches of sciences and mathematics. Some of them merely take advantage of the compact representation of a set of numbers in a matrix. These numbers often come from real problems but due to the rounding errors and measurement errors, the measured numbers often are accompanied by uncertainty. Mathematically, instead of working with an uncertain number, one can works with the ends of an interval which contains that number and so using the interval matrices give a more realistic description of the measured results.

Hansen and smith started the use of interval arithmetic in matrix computations and after that many other authors have studied interval matrices [20].

The systems of linear interval equations are very common in several areas of engineering and sciences. These systems frequently arise in practice, especially in situations where the data can not be measured exactly but are known to be in a certain range. Finding the solution of interval linear systems is a challenging problem in interval arithmetic. Methods for solving interval linear systems have been developed in the literature. Alefeld and Mayer [1] presented an overview on applications of interval arithmetic and discussed verification methods for linear systems of equations. For more research works about this topic, we refer the interested reader to [2, 23, 9, 7, 8, 22, 14, 5, 6, 25, 3, 15, 21, 16, 11, 4].

The LU decomposition transforms a square real matrix $A$ as the product of a lower triangular matrix $L$ and an upper triangular matrix $U$ satisfying $A = LU$. This decomposition usually is used for solving square systems of linear equations, inverting a matrix or computing the determinant of a matrix.

The LU decomposition can be considered as the matrix form of Gaussian elimination. It has been generalized to interval matrices, see [18, 17]. Interval Gauss elimination in classical interval arithmetic yields the LU decomposition of the interval matrix $\mathbf{A}$. But this decomposition does not satisfy the relation $\mathbf{A} = \mathbf{LU}$ and only satisfies the weaker relation $\mathbf{A} \subseteq \mathbf{LU}$. In this paper we present the LU decomposition of a generalized interval matrix $\mathbf{A}$ such that it satisfies $\mathbf{A} = \mathbf{LU}$ under modified interval arithmetic. Then we use this LU decomposition to obtain the exact solution of the interval linear systems based on modified interval arithmetic.

The rest of the paper is organized as follows: Section 2 gives an overview of the generalized intervals and modified interval arithmetic. Section 3 presents the modified interval LU decomposition and Section 4 shows how it can be used to compute the exact solution of an interval linear system based on the modified interval arithmetic. Some numerical results are given in Section 5 to show the effectiveness of the new approach. Finally in Section 6, we complete the paper by a brief concluding remark.

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## 2. Preliminaries

In this section we present some definitions and notation which are useful in our further considerations.

In this paper, ordinary and boldface letters, respectively, stand for the real and interval quantities. The field of real numbers is shown by $\mathbb{R}$. $\mathbb{IR}$ stands for the set of real interval numbers. The space of $n$-dimensional interval real vectors and $m$-by-$n$ interval real matrices are shown by $\mathbb{IR}^n$ and $\mathbb{IR}^{m\times n}$, respectively. For an interval number $\mathbf{x} = [\underline{x}, \overline{x}] \in \mathbb{IR}$, the midpoint of $\mathbf{x}$ is defined as $m(\mathbf{x}) = \frac{\overline{x}+\underline{x}}{2}$. Also the radius of $\mathbf{x}$ is $rad(\mathbf{x}) = \frac{\overline{x}-\underline{x}}{2}$. The set of proper intervals $\mathbb{IR} = \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \leq \overline{\mathbf{x}},\ \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$, is extended by the set $\overline{\mathbb{IR}} := \{\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] : \underline{\mathbf{x}} \geq \overline{\mathbf{x}},\ \underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}\}$ of improper intervals, obtaining the set of generalized intervals $\mathbb{KR} = \{\mathbf{a} = [\underline{\mathbf{a}}, \overline{\mathbf{a}}] : \underline{\mathbf{a}}, \overline{\mathbf{a}} \in \mathbb{R}\}$. For example $[-1, 1]$ and $[1, -1]$ are generalized intervals. For an interval matrix $\mathbf{A} \in \mathbb{KR}^{m\times n}$, $m(\mathbf{A})$ and $rad(\mathbf{A})$ are defined componentwise.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{KR}$ and $\circledast \in \{+, -, *, /\}$. The modified interval operation $\circledast$ introduced in [20] is defined as

$$\mathbf{x} \circledast \mathbf{y} = [m(\mathbf{x}) \circledast m(\mathbf{y}) - k, m(\mathbf{x}) \circledast m(\mathbf{y}) + k],$$

therein $k = \min\{(m(\mathbf{x}) \circledast m(\mathbf{y})) - \alpha, \beta - (m(\mathbf{x}) \circledast m(\mathbf{y}))\}$ and $\alpha, \beta$ are the end points of the interval $\mathbf{x} \odot \mathbf{y}$, in which $\odot$ denotes the corresponding operation in the classical interval arithmetic. In particular

$$\mathbf{x} + \mathbf{y} = [(m(\mathbf{x}) + m(\mathbf{y})) - k, (m(\mathbf{x}) + m(\mathbf{y})) + k], \qquad k = \frac{(\overline{\mathbf{x}} + \overline{\mathbf{y}}) - (\underline{\mathbf{x}} + \underline{\mathbf{y}})}{2}.$$

$$\mathbf{x} - \mathbf{y} = [(m(\mathbf{x}) - m(\mathbf{y})) - k, (m(\mathbf{x}) - m(\mathbf{y})) + k], \qquad k = \frac{(\overline{\mathbf{x}} + \overline{\mathbf{y}}) - (\underline{\mathbf{x}} + \underline{\mathbf{y}})}{2}.$$

$$1/\mathbf{x} = [\frac{1}{m(\mathbf{x})} - k, \frac{1}{m(\mathbf{x})} + k], \qquad k = \min\{\frac{1}{\overline{\mathbf{x}}}(\frac{\overline{\mathbf{x}} - \underline{\mathbf{x}}}{\overline{\mathbf{x}} + \underline{\mathbf{x}}}), \frac{1}{\underline{\mathbf{x}}}(\frac{\overline{\mathbf{x}} - \underline{\mathbf{x}}}{\overline{\mathbf{x}} + \underline{\mathbf{x}}})\}, 0 \notin [\underline{\mathbf{x}}, \overline{\mathbf{x}}].$$

$$\mathbf{x} * \mathbf{y} = [m(\mathbf{x})m(\mathbf{y}) - k, m(\mathbf{x})m(\mathbf{y}) + k], \qquad k = \min\{(m(\mathbf{x})m(\mathbf{y})) - \alpha, \beta - (m(\mathbf{x})m(\mathbf{y}))\},$$

therein $\alpha = \min\{\underline{\mathbf{x}}\underline{\mathbf{y}}, \underline{\mathbf{x}}\overline{\mathbf{y}}, \overline{\mathbf{x}}\underline{\mathbf{y}}, \overline{\mathbf{x}}\overline{\mathbf{y}}\}$ and $\beta = \max\{\underline{\mathbf{x}}\underline{\mathbf{y}}, \underline{\mathbf{x}}\overline{\mathbf{y}}, \overline{\mathbf{x}}\underline{\mathbf{y}}, \overline{\mathbf{x}}\overline{\mathbf{y}}\}$.

It is to be noted that in the modified interval arithmetic, we have $\mathbf{x} - \mathbf{x} = [0, 0]$ and $\mathbf{x}/\mathbf{x} = [1, 1]$, while in the classical interval arithmetic we just have $0 \in \mathbf{x} - \mathbf{x}$ and $1 \in \mathbf{x}/\mathbf{x}$. Also, if $\odot$ and $\circledast$ stand for the classical interval arithmetic and modified interval arithmetic, respectively, then for $\mathbf{x}, \mathbf{y} \in \mathbb{KR}$ we have

$$\mathbf{x} \circledast \mathbf{y} \subseteq \mathbf{x} \odot \mathbf{y}.$$

Note that by the introduced operations in the modified interval arithmetic, the set of generalized intervals possesses group properties with respect to addition and multiplication. So, many important laws such as distributive law which does not hold in the classical interval arithmetic, for generalized intervals holds.

Similar to the convention used in [20], two intervals $\mathbf{x}, \mathbf{y} \in \mathbb{KR}$ are said to be equivalent when $m(\mathbf{x}) = m(\mathbf{y})$ and is denoted by $\mathbf{x} \approx \mathbf{y}$. In particular when $m(\mathbf{x}) = m(\mathbf{y})$ and $rad(\mathbf{x}) = rad(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$. If $m(\mathbf{x}) = 0$ then we say that $\mathbf{x}$ is a zero interval number. We use a similar convention for generalized interval matrices.

## 3. Modified interval LU decomposition

As we mentioned above, the modified interval arithmetic is a group for addition and multiplication and thus this property allows us to construct an LU decomposition of a generalized interval matrix $\mathbf{A}$ such that $\mathbf{A} = \mathbf{LU}$. Our approach is similar to the one presented in [12], with this difference that we are using a different interval arithmetic introduced in Section 2, namely modified interval arithmetic.

In order to obtain a modified interval LU decomposition of the generalized interval matrix $\mathbf{A} \in \mathbb{KR}^{n\times n}$, similar to the case in real numbers and classical intervals, the Gauss elimination algorithm based on modified interval arithmetic must be applied. This leads to a modified interval LU decomposition of the generalized interval matrix $\mathbf{A}$. This modified interval LU decomposition can be formulated in the following way

$$\mathbf{L}_{ii} = 1, \qquad \text{and} \qquad \mathbf{L}_{ij} = 0, \qquad \text{for} \quad i < j, \tag{1}$$

$$\mathbf{L}_{ij} = \frac{\mathbf{A}_{ij} - \sum_{k=1}^{j-1} \mathbf{L}_{ik}\mathbf{U}_{kj}}{\mathbf{U}_{jj}}, \qquad \text{for} \quad j < i, \tag{2}$$

$$\mathbf{U}_{ij} = 0, \qquad \text{for} \quad i > j, \tag{3}$$

$$\mathbf{U}_{ij} = \mathbf{A}_{ij} - \sum_{k=1}^{i-1} \mathbf{L}_{ik}\mathbf{U}_{kj}, \qquad \text{for} \quad i \leq j. \tag{4}$$

These expressions allow constructing a modified interval LU decomposition in a recursive way.

  

**Theorem 1** *Let* $\mathbf{A} \in \mathbb{KR}^{n\times n}$ *be a generalized interval matrix. Provided that the interval matrices* $\mathbf{L}$ *and* $\mathbf{U}$ *defined by* (1)-(4) *can be constructed, then we have* $\mathbf{A} \approx \mathbf{LU}$.

**Proof.** The proof is similar to the proof of Proposition 1 in [12] with this difference that the employed interval arithmetic here is different from that one used in [12]. ☐

**Example 1** *[12] Consider the interval matrix*

$$\mathbf{A} = \begin{pmatrix} [9,11] & [-1,1] & [-1,1] \\ [-11,11] & [8,12] & [-2,2] \\ [-11,11] & [-12,12] & [7,13] \end{pmatrix}.$$

*For computing modified LU decomposition, using relations* (1)-(4) *we obtain*

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ [-1.2001, 1.2001] & 1 & 0 \\ [-1.2001, 1.2001] & [-1.6400, 1.6400] & 1 \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} [9.0000, 11.0000] & [-1.0000, 1.0000] & [-1.0000, 1.0000] \\ 0 & [6.8000, 13.2000] & [-3.2001, 3.2001] \\ 0 & 0 & [0.5519, 19.4481] \end{pmatrix},$$

*which satisfy* $\mathbf{A} \approx \mathbf{LU}$.

**Example 2** *[12] Consider the interval matrix*

$$\mathbf{A} = \begin{pmatrix} [2,3] & 1 & 0 \\ 1 & [2,3] & 1 \\ 0 & 1 & [2,3] \end{pmatrix}.$$

*The modified interval LU decomposition gives the following interval triangular matrices*

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ [0.3333, 0.4667] & 1 & 0 \\ [0.0000, 0.0000] & [0.3749, 0.5774] & 1 \end{pmatrix},$$

$$\mathbf{U} = \begin{pmatrix} [2.0000, 3.0000] & [1.0000, 1.0000] & [0.0000, 0.0000] \\ 0 & [1.5333, 2.6667] & [1.0000, 1.0000] \\ 0 & 0 & [1.4226, 2.6251] \end{pmatrix},$$

*these interval matrices satisfy* $\mathbf{A} \approx \mathbf{LU}$.

In real arithmetic, LU decomposition is a useful tool for evaluating the inverse of a matrix and computing its determinant. However, these two applications do not work in classical interval arithmetic since many important properties of real numbers do not hold for classical interval numbers. But due to the smooth behavior of the generalized interval numbers under the introduced modified interval arithmetic, some substantial properties hold in the new arithmetic.

## 4. Interval linear systems

In this section, we want to consider the interval system of linear equations

$$\mathbf{A}x \approx \mathbf{b},$$

which is a system of $n$ linear equations in $n$ variables whose coefficients and right-hand side vary within some real intervals. In literature, almost an estimation of the solution set of an interval system is obtained and finding the exact solution is so heavy. Here, we obtain the exact solution of an interval linear system under the modified interval arithmetic. We do this using our proposed modified LU factorization.

First, we consider the interval triangular linear systems. An interval triangular linear system is an interval linear system in which the coefficient matrix is triangular. Let $\mathbf{A}x \approx \mathbf{b}$ be an interval triangular linear system in which

  

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & 0 & \cdots & 0\\ \vdots & \mathbf{A}_{22} & \ddots & \vdots\\ \vdots & & \ddots & 0\\ \mathbf{A}_{n1} & \cdots & \cdots & \mathbf{A}_{nn}\end{pmatrix},\qquad \mathbf{b}=\begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \vdots\\ \mathbf{b}_n\end{pmatrix},$$

and $0\notin\mathbf{A}_{ii}$. By forward substitution and using opposite and inverse of a generalized interval, the following result is obtained

$$x_i=\frac{\mathbf{b}_i-\sum_{j=1}^{i-1}\mathbf{A}_{ij}x_j}{\mathbf{A}_{ii}},\qquad i=1,\dots,n. \tag{5}$$

For a triangular linear interval system with upper triangular coefficient matrix, the backward substitution technique is used. In fact if in the interval linear system $\mathbf{A}x\approx\mathbf{b}$, $\mathbf{A}\in\mathbb{KR}^{n\times n}$ is upper triangular and

$$\mathbf{A}=\begin{pmatrix}\mathbf{A}_{11} & \cdots & \cdots & \mathbf{A}_{1n}\\ 0 & \mathbf{A}_{22} & & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & \mathbf{A}_{nn}\end{pmatrix},\qquad \mathbf{b}=\begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\\ \vdots\\ \mathbf{b}_n\end{pmatrix},$$

where $0\notin\mathbf{A}_{ii}$, then by backward substitution we obtain the following result

$$x_i=\frac{\mathbf{b}_i-\sum_{j=i+1}^{n}\mathbf{A}_{ij}x_j}{\mathbf{A}_{ii}},\qquad i=1,\dots,n. \tag{6}$$

**Proposition 1** *Let $\mathbf{A}x\approx\mathbf{b}$ be an interval triangular linear system with lower (upper) triangular coefficient matrix $\mathbf{A}\in\mathbb{KR}^{n\times n}$ and right-hand side $\mathbf{b}\in\mathbb{KR}^n$, then $\mathbf{x}=(\mathbf{x}_1,\dots,\mathbf{x}_n)^T\in\mathbb{KR}^n$ constructed by Eq. (5) (Eq. (6)) satisfies $\mathbf{Ax}\approx\mathbf{b}$.*

**Proof.** For this purpose, it is sufficient that we prove $(\mathbf{Ax})_i\approx\mathbf{b}_i$. But we have

$$\begin{aligned}(\mathbf{Ax})_i &\approx \sum_{k=1}^{i}\mathbf{A}_{ik}\mathbf{x}_k \approx \sum_{k=1}^{i}\mathbf{A}_{ik}\frac{\mathbf{b}_k-\sum_{j=1}^{k-1}\mathbf{A}_{kj}\mathbf{x}_j}{\mathbf{A}_{kk}}\\ &\approx \sum_{k=1}^{i-1}\mathbf{A}_{ik}\frac{\mathbf{b}_k-\sum_{j=1}^{k-1}\mathbf{A}_{kj}\mathbf{x}_j}{\mathbf{A}_{kk}}+(\mathbf{b}_i-\sum_{j=1}^{i-1}\mathbf{A}_{ij}\mathbf{x}_j)\\ &\approx \sum_{j=1}^{i-1}\mathbf{A}_{ij}\mathbf{x}_j+(\mathbf{b}_i-\sum_{j=1}^{i-1}\mathbf{A}_{ij}\mathbf{x}_j)\approx\mathbf{b}_i.\end{aligned}$$

For the case where the coefficient matrix is upper triangular, the proof is similar. □

*4.1. Solving an interval linear system*

Given a system of linear equations involving the generalized interval numbers

$$\mathbf{A}x\approx\mathbf{b}, \tag{7}$$

we want to obtain the exact solution of this equation under the modified interval arithmetic. Using the LU decomposition of $\mathbf{A}$, Eq. (7) can be written equivalently as

$$(\mathbf{LU})x\approx\mathbf{b}. \tag{8}$$

By associative law of multiplication for interval matrices under modified interval arithmetic, the Eq. (8) is equivalent to

$$\mathbf{L}(\mathbf{U}x)\approx\mathbf{b}. \tag{9}$$

Now to solve the system (7) using LU decomposition, the following two steps should be done:

**1.** First, using relation (5) solve the equation $\mathbf{L}y\approx\mathbf{b}$ for $y$ and obtain $\mathbf{y}$.
**2.** Second, using relation (6) solve the equation $\mathbf{U}x\approx\mathbf{y}$ for $x$ and obtain the solution $\mathbf{x}$.

By the above steps, the exact solution of the system (7) is obtained and so we can present the following theorem.

  

**Theorem 2** *Let* $\mathbf{A}x \approx \mathbf{b}$ *be a system of interval linear equations. If the generalized interval vector* $\mathbf{x}$ *is obtained from two above steps, then it solves the system* $\mathbf{A}x \approx \mathbf{b}$.

Note that associative law for multiplication of the interval matrices does not hold in classical interval arithmetic and so the above steps can not be executed in classical case. Also, in classical interval arithmetic, the solution set of an interval linear system often is very complicated and the researchers often are interested in finding outer or inner approximations for this solution set. In fact the obtained solution set is not exact and only we have an inclusion relation. But the new approach presented in the current paper gives the exact solution of an interval linear system under modified interval arithmetic.

## 5. Numerical results

In this section, we use three test problems taken from the literature [20, 10, 19] to show the effectiveness of the new approach for solving interval linear systems.

**Example 3** *[20, 19] Consider the system of interval linear equations* $\mathbf{A}x \approx \mathbf{b}$ *with*

$$\mathbf{A} = \begin{pmatrix} [3.7, 4.3] & [-1.5, -0.5] & [0, 0] \\ [-1.5, -0.5] & [3.7, 4.3] & [-1.5, -0.5] \\ [0, 0] & [-1.5, -0.5] & [3.7, 4.3] \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} [-14, 14] \\ [-9, 9] \\ [-3, 3] \end{pmatrix}.$$

*By the new method, we obtain the following solution set*

$$\mathbf{x} = \begin{pmatrix} [-5.9418, 5.9418] \\ [-5.4781, 5.4781] \\ [-2.6907, 2.6907] \end{pmatrix}.$$

*Note that* $\mathbf{x}$ *satisfies* $\mathbf{Ax} \approx \mathbf{b}$ *which confirms Theorem 2.*

*Nirmala et al. [20], using the inverse interval matrix in modified interval arithmetic obtained the following solution*

$$\begin{pmatrix} [-6.988, 6.988] \\ [-5.981, 5.981] \\ [-3.185, 3.185] \end{pmatrix}.$$

*Using interval Gaussian elimination, Ning and Kearfott [19] obtained the same solution set*

$$\begin{pmatrix} [-6.38, 6.38] \\ [-6.40, 6.40] \\ [-3.40, 3.40] \end{pmatrix}.$$

*As one can see, the solution set obtained by our method is sharper and so better than the other obtained results.*

**Example 4** *[20, 10, 19] Let us consider the system of interval linear equations* $\mathbf{A}x \approx \mathbf{b}$ *with*

$$\mathbf{A} = \begin{pmatrix} [3.7, 4.3] & [-1.5, -0.5] & [0, 0] \\ [-1.5, -0.5] & [3.7, 4.3] & [-1.5, -0.5] \\ [0, 0] & [-1.5, -0.5] & [3.7, 4.3] \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} [-14, 0] \\ [-9, 0] \\ [-3, 0] \end{pmatrix}.$$

*Using the new approach, we obtain the solution set*

$$\mathbf{x} = \begin{pmatrix} [-4.4465, 0] \\ [-3.7858, 0] \\ [-1.6965, 0] \end{pmatrix},$$

*which satisfies* $\mathbf{Ax} \approx \mathbf{b}$.

*Using the inverse interval matrix technique, the following result is obtained by Nirmala et al. [20]*

$$\begin{pmatrix} [-4.482, 0] \\ [-3.816, 0] \\ [-1.776, 0.066] \end{pmatrix}.$$

*Using interval Gaussian elimination method, Ning and Kearfott [19] obtained the following result*

$$\begin{pmatrix} [-6.38, 0] \\ [-6.40, 0] \\ [-3.40, 0] \end{pmatrix}.$$

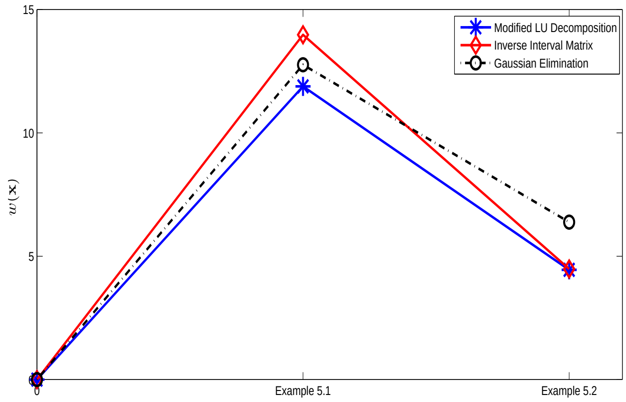


Figure 1. The width of the obtained solution sets by three methods for Examples 3 and 4

*Using Hansen's technique [13] or Rohn's reformulation of [24], Ning and Kearfott [19] obtained the solution set*

$$\begin{pmatrix} [-6.38, 1.12] \\ [-6.40, 1.54] \\ [-3.40, 1.40] \end{pmatrix}.$$

*Also, Ning and Kearfott [19] obtained the solution set*

$$\begin{pmatrix} [-6.38, 1.67] \\ [-6.40, 2.77] \\ [-3.40, 2.40] \end{pmatrix}.$$

*As one can see, the solution set obtained by our approach is better than the one obtained by the other techniques.*

In Fig. 1, the width of the obtained solution sets by three methods for Examples 3 and 4 are shown. As one can see, the new approach yields better results than the other methods.

**Example 5** *Consider the system of interval linear equations* $\mathbf{A}x \approx \mathbf{b}$ *in which*

$$\mathbf{A} = \begin{pmatrix} [-1, 2] & [2, 3] & [3, 4] \\ [2, 5] & [-3, 1] & [4, 6] \\ [-2, 0] & [1, 3] & [3, 4] \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} [1, 2] \\ [3, 4] \\ [5, 6] \end{pmatrix}.$$

*The proposed technique in [20], i.e., using the modified inverse of* $\mathbf{A}$ *is not applicable here, because according to the mentioned definitions and theorems in that paper,* $\mathbf{A}$ *is not invertible (in fact* $0 \in det(\mathbf{A}) = [-122.7501, 65]$ *and so* $\mathbf{A}$ *is singular).*

*Now we want to use the new approach in this paper. If we apply the LU decomposition for* $\mathbf{A}$ *and then use the Eqs. (5) and (6), we obtain the solution set*

$$\mathbf{x} = 10^3 \begin{pmatrix} [-4.6808, 4.6765] \\ [-0.2791, 0.2759] \\ [-0.0245, 0.0282] \end{pmatrix},$$

*which satisfies* $\mathbf{Ax} \approx \mathbf{b}$.

  

## 6. Concluding remarks

In this paper, we presented a modified interval LU decomposition in the framework of the generalized intervals under modified interval arithmetic introduced in [20]. Because of group structure of this arithmetic, the new approach for LU decomposition of a generalized interval matrix $\mathbf{A}$, satisfies $\mathbf{A} = \mathbf{LU}$. Also by the presented analysis in Section 4, we proved that it can be used to construct the exact solution of an interval linear system of equations under modified interval arithmetic. By some examples, we showed the feasibility and effectiveness of the new approach proposed in the current paper.

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