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A computational method to solve fractional-order Fokker-Planck equations based on Touchard polynomials

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This manuscript presents a new approximation method for fractional-order Fokker-Planck equations based on Touchard polynomial approximation. We provide new Caputo and extra Caputo pseudo-operational matrices for these polynomials. Then, utilizing mentioned pseudo-operational matrices and an optimal method, the considered equation leads to a system of algebraic equations which can be solved by mathematical software. Finally, we illustrate the advantages of the suggested technique through several numerical examples. Copyright © 2022 Shahid Beheshti University.

Keywords: Fractional-order Fokker-Planck equations; Touchard polynomials; Pseudo-operational matrix.

1. Introduction

In this study, we focus on the following nonlinear time-space fractional-order Fokker-Planck equations [16]

$$D_t^{\alpha} z(x,t) = \left(-D_x^{\beta}\mathfrak{A}(x,t,z) + D_x^{2\beta}\mathfrak{B}(x,t,z)\right) z(x,t), \tag{1}$$

where $(x,t) \in [0,1] \times [0,1]$, and $0 < \alpha, \beta \leq 1$. A backward Kolmogorov equation is a kind of fractional-order partial differential equation as follows

$$D_t^{\alpha} z(x,t) = \left(-\mathfrak{A}(x,t)D_x^{\beta} + \mathfrak{B}(x,t)D_x^{2\beta}\right) z(x,t), \tag{2}$$

subject to the initial condition $z(x,0) = \mu(x)$, where $\mu(x)$ is a known function, $\mathfrak{A}(x,t,z) > 0$ is a drift coefficient, $\mathfrak{B}(x,t,z) > 0$ is diffusion coefficient, $z(x,t)$ is an unknown function. α, β are denoted the Caputo fractional derivative, which is defined in Ref. [11].

The classical form of nonlinear Fokker-Planck equations is achieved by choosing $\alpha = \beta = 1$ in Eq. (1).

Fokker-Planck equations are a class of important equations in mathematics that appear in the modeling of various phenomena in science and engineering such as polymeric networks, wave propagation, tumor growth, continuous random walk, neuroscience, nonlinear hydrodynamics, biophysics, DNA and RNA polymerases, marketing and psychology [8, 24, 25, 22]. These equations are an important class of partial differential equations. These equations are an important class of partial differential equations. Partial differential equations have wide applications that various computational methods to solve different kinds of these were developed such as generalized pseudospectral method [3], fractional Lagrande function method [5], general Lagrange scaling function method [18], a hybrid numerical method [4], and so on.

Actually, the Fokker-Planck equation demonstrates the variation of the probability of a stochastic function in space and time. Hence it is naturally used to describe the transport of solutes. The Fokker-Planck equation was first used by Fokker and Planck [17] to explain the Brownian motion of particles.

Recently, a considerable amount of papers have proposed methods for solving the integer or non-integer order Fokker-Planck equations. Tatari et al solved integer-order these equations using Adomian decomposition method [22]. The variational iteration method was used to solve the considered problem [12]. Furthermore, Saravanan and Magesh [20] presented a fractional variational method to solve the linear and nonlinear fractional Fokker-Planck partial differential equations. The finite difference method and

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Fourier analysis were applied for solving the time-space fractional Fokker-Planck equation with a variable force field and diffusion [15]. Also, the Cubic B-Spline scaling function method proposed in Ref. [10] to solve nonlinear Fokker-Planck equations. In [13], the generalized Lagrange Jacobi Gauss-Lobatto collocation method applied to solve linear and nonlinear Fokker-Planck equations

On the other hand, Touchard polynomials are introduced by Jacques Touchard in 1939 [23]. Sabermahani et al. [11, 19] presented Touchard and general Touchard wavelets to find the numerical solutions of two different kinds of fractional problems.

Here, we present an approximation method based on Touchard polynomials. The properties of these polynomials with the help of designing an optimal method, are the main tools to present the mentioned technique. To do this, new Caputo and extra Caputo pseudo-operational matrices for Touchard polynomials are proposed. In addition, the process of obtaining these matrices has a positive effect on reducing the error in the calculation.

Structure of the manuscript

In this paper, our main aim is to modify the computational method to solve fractional-order Fokker-Planck equations, numerically. This paper is organized as follows. In Section 2, we recall the definition of Touchard polynomials and their properties. In Section 3, we present new achievements of these polynomials. We present Caputo and extra Caputo pseudo-operational matrices, in this section. In Section 4, we propose a computational method for solving considered equations. Finally, Section 5 will give a conclusion briefly.

2. Touchard polynomials

Touchard polynomials are also called exponential polynomials, are defined by

$$T_m(x) = e^{-x}\sum_{k=0}^{\infty}\frac{k^m x^k}{k!} = e^{-x}\left(x\frac{d}{dx}\right)^m e^x.$$

These polynomials $T_m(x)$ are defined by means of the exponential generating functions in the following form

$$e^{x(e^t-1)} = \sum_{m=0}^{\infty} T_m(x)\frac{t^m}{m!}. \tag{3}$$

In addition, the Touchard polynomials of degree m are defined as [9]

$$\begin{aligned} T_m(x) &= \sum_{k=0}^{m}\left\{\begin{matrix} m \\ k \end{matrix}\right\} x^k \\ &= \sum_{k=0}^{m}\left(\sum_{i=0}^{k}\frac{(-1)^i}{k!}\left(\begin{matrix} m \\ k \end{matrix}\right)(k-i)^m x^k\right), \end{aligned} \tag{4}$$

where $\left\{\begin{matrix} m \\ k \end{matrix}\right\}$ is the Stirling number of second kind.

Theorem 1 *For $m \geq 0$, the Touchard polynomials $T_m(x)$ can be calculated by [14]*

$$\sum_{k=0}^{m}\left\{\begin{matrix} m \\ k \end{matrix}\right\} T_k(x) = x^m.$$

To see the different features of these polynomials, Refs. [14, 2, 1] are suggested.

A function g defined over $[0, 1]$ may be expressed in terms of the Touchard polynomials as

$$g(x) \approx \sum_{m=0}^{M-1} g_m T_m(x) = G^T\Theta(x), \tag{5}$$

the vector coefficient G is given by

$$G = D^{-1}\langle g(x), \Theta(x)\rangle, \quad D = \langle \Theta(x), \Theta(x)\rangle,$$

in which

$$G = [g_0, g_1, \cdots, g_{M-1}]^T, \quad \Theta(x) = [T_0(x), T_1(x), \cdots, T_{M-1}(x)]^T. \tag{6}$$

Similarly, suppose that $g(x, t)$ be an arbitrary element of $L^2([0, 1] \times [0, 1])$, we can approximate this function using Touchard polynomials as

$$g(x, t) \approx \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} u_{mn} T_m(x) T_n(t) = \Theta^T(x) U \Theta(t), \tag{7}$$

where the matrix coefficient U is derived as

$$U = D^{-1}\langle\langle g(x,t), \Theta(x)\rangle, \Theta(t)\rangle \tilde{D}^{-1}, \quad \tilde{D} = \langle\Theta(t), \Theta(t)\rangle.$$

3. New achievements

The object of this section is to introduce the Caputo and extra Caputo pseudo-operational matrices using Caputo fractional derivative and Touchard polynomials properties.

3.1. Caputo pseudo-operational matrix

Here, we construct the Caputo pseudo-operational matrix for the considered polynomials in the following form

$$D_x^\alpha \Theta(x) \approx \zeta(\alpha, x)\Theta(x), \tag{8}$$

where $\zeta(\alpha, x) = x^{-\alpha}\eta$ is a pseudo-operational matrix of the fractional derivative with respect to x, which is computed using Eq. (3) and the property of Caputo fractional derivative, from the following process:

$$\begin{aligned} D_x^\alpha T_m(x) &= D_x^\alpha \left(\sum_{k=0}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} x^k \right) \\ &= \sum_{k=0}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} D_x^\alpha \left(x^k \right) \\ &= \sum_{k=0}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k-\alpha} \\ &= x^{-\alpha} \sum_{k=0}^{m} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} x^{k} \\ &= x^{-\alpha} \sum_{k=0}^{m} c_{m,k}^\alpha x^k, \end{aligned} \tag{9}$$

where $c_{m,k}^\alpha = \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)}$. Now, we can expand x^k via the Touchard polynomials, then we get

$$x^k \approx \sum_{j=0}^{M-1} \hat{u}_{j,k} T_j(x). \tag{10}$$

Thus, we obtain

$$\begin{aligned} D_x^\alpha T_m(x) &\approx x^{-\alpha} \sum_{j=0}^{M-1} \sum_{k=0}^{m} c_{m,k}^\alpha \hat{u}_{j,k} T_j(x) \\ &= x^{-\alpha} \sum_{j=0}^{M-1} \sum_{k=0}^{m} \hat{c}_{m,k,j} T_j(x), \end{aligned} \tag{11}$$

where $\hat{c}_{m,k,j} = c_{m,k}^\alpha \hat{u}_{j,k}$. As a result

$$D_x^\alpha T_m(x) \approx x^{-\alpha} \sum_{j=0}^{M-1} \eta_{m,k,j} T_j(x), \quad \eta_{m,k,j} = \sum_{k=0}^{m} \hat{c}_{m,k,j}.$$

For example, for $M = 3, \alpha = \frac{1}{2}$, then we have

$$\eta = \begin{bmatrix} \frac{1}{\sqrt{\pi}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{\pi}} & 0 \\ 0 & -\frac{2}{3\sqrt{\pi}} & \frac{8}{3\sqrt{\pi}} \end{bmatrix}.$$

3.2. Extra Caputo pseudo-operational matrix

In this subsection, we present an extra Caputo pseudo-operational matrix for the Touchard polynomials as follows

$$D^{\alpha}\left(x^{\rho}\Theta(x)\right)\approx\varphi(x,\alpha,\rho)\Theta(x),\quad \rho=0,1,2,\cdots, \tag{12}$$

$\varphi(x,\alpha,\rho)$ is called the extra Caputo pseudo-operational matrix. Using Eq. (3) and the property of Caputo fractional derivative, each component of the presented matrix is calculated as follows

$$\begin{aligned} D_x^{\alpha}\left(x^{\rho}T_m(x)\right) &= D_x^{\alpha}\left(\sum_{k=0}^{m}\left\{\begin{array}{c} m \\ k \end{array}\right\}x^{k+\rho}\right) \\ &= \sum_{k=0}^{m}\left\{\begin{array}{c} m \\ k \end{array}\right\}D_x^{\alpha}\left(x^{k+\rho}\right) \\ &= \sum_{k=0}^{m}\left\{\begin{array}{c} m \\ k \end{array}\right\}\frac{\Gamma(k+\rho+1)}{\Gamma(k+\rho+1-\alpha)}x^{k+\rho-\alpha} \\ &= x^{\rho-\alpha}\sum_{k=0}^{m}\left\{\begin{array}{c} m \\ k \end{array}\right\}\frac{\Gamma(k+\rho+1)}{\Gamma(k+\rho+1-\alpha)}x^{k} \\ &= x^{\rho-\alpha}\sum_{k=0}^{m}d_{m,k}^{\alpha}x^{k}, \end{aligned} \tag{13}$$

approximating x^k via Eq. (10), we insert it in the above equation, then we obtain

$$\begin{aligned} D_x^{\alpha}\left(x^{\rho}T_m(x)\right) &\approx x^{\rho-\alpha}\sum_{j=0}^{M-1}\sum_{k=0}^{m}d_{m,k}^{\alpha,\rho}\hat{u}_{j,k}T_j(x) \\ &= x^{\rho-\alpha}\sum_{j=0}^{M-1}\sum_{k=0}^{m}\hat{d}_{m,k,j}^{\alpha,\rho}T_j(x), \end{aligned} \tag{14}$$

where $\hat{d}_{m,k,j}^{\alpha,\rho}=d_{m,k}^{\alpha,\rho}\hat{u}_{j,k}$. Hence, each of the rows of the mentioned matrix is computed as

$$D_x^{\alpha}\left(x^{\rho}T_m(x)\right)\approx x^{\rho-\alpha}\left[\sum_{k=0}^{m}\hat{d}_{m,k,0}^{\alpha,\rho},\sum_{k=0}^{m}\hat{d}_{m,k,1}^{\alpha,\rho},\cdots,\sum_{k=0}^{m}\hat{d}_{m,k,M-1}^{\alpha,\rho}\right]\Theta(x),\quad m=0,1,\cdots,M-1.$$

4. Explanation of the method

To solve the considered problems in Eqs. (1) and (2), we approximate the function

$$z(x,t)\approx\Theta^{T}(x)Z\hat{\Theta}(t), \tag{15}$$

where $\Theta^T(x)$ defined in Eq. (6) and equals to $\hat{\Theta}(t)=[T_0(t),T_1(t),\cdots,T_{N-1}(t)]^T$.

By utilizing Caputo pseudo-operational matrix, we obtain the following relations:

$$D_t^{\alpha}z(x,t)\approx\Theta^{T}(x)Z\zeta(\alpha,t)\hat{\Theta}(t), \tag{16}$$

$$D_x^{\beta}z(x,t)\approx\Theta^{T}(x)\zeta^{T}(\beta,x)Z\hat{\Theta}(t), \tag{17}$$

and

$$D_x^{2\beta}z(x,t)\approx\Theta^{T}(x)\zeta^{T}(2\beta,x)Z\hat{\Theta}(t), \tag{18}$$

Now, due to Eq. (1), we approximate the other components of the problem. To do this, for the special case of $\mathfrak{A}(x,t,z)$ and $\mathfrak{B}(x,t,z)$, we consider

$$\mathfrak{A}(x,t,z)=\sum_{r=1}^{s_1}\mathfrak{H}_r(x)\mathfrak{S}_r(t)\mathfrak{F}_r(z),\quad \mathfrak{B}(x,t,z)=\sum_{s=1}^{s_2}\mathfrak{K}_s(x)\mathfrak{V}_s(t)\mathfrak{T}_s(z), \tag{19}$$

subject to $\mathfrak{H}_r(x),\mathfrak{S}_r(t),\mathfrak{F}_r(z),\mathfrak{K}_s(x),\mathfrak{V}_s(t)$ and $\mathfrak{T}_s(z)$ are analyical functions. So, we get

$$\mathfrak{H}_r(x)=\sum_{i=0}^{\infty}h_{ir}x^{i},\quad \mathfrak{F}_r(z)=\sum_{j=0}^{\infty}f_{jr}z^{j},\quad \mathfrak{K}_s(x)=\sum_{l=0}^{\infty}a_{ls}x^{s},\quad \mathfrak{T}_s(z)=\sum_{w=0}^{\infty}v_{ws}z^{w}. \tag{20}$$

Due to aforesaid equations, we derive

$$\begin{aligned}\mathfrak{A}(x,t,z)z(x,t) &\approx \sum_{r=1}^{s_1}\mathfrak{H}_r(x)\mathfrak{S}_r(t)\mathfrak{F}_r(z)z(x,t) \\ &= \sum_{r=1}^{s_1}\left[\sum_{i=0}^{\infty}h_{ir}x^i\right]\mathfrak{S}_r(t)\left[\sum_{j=0}^{\infty}f_{jr}z^j\right]z(x,t) \\ &= \sum_{r=1}^{s_1}\mathfrak{S}_r(t)\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}h_{ir}f_{jr}x^iz^{j+1}\right],\end{aligned} \tag{21}$$

and

$$\begin{aligned}\mathfrak{B}(x,t,z)z(x,t) &\approx \sum_{s=1}^{s_2}\mathfrak{K}_s(x)\mathfrak{V}_s(t)\mathfrak{T}_s(z)z(x,t) \\ &= \sum_{s=1}^{s_2}\left[\sum_{l=0}^{\infty}a_{ls}x^s\right]\mathfrak{V}_s(t)\left[\sum_{w=0}^{\infty}v_{ws}z^w\right]z(x,t) \\ &= \sum_{s=1}^{s_2}\mathfrak{V}_s(t)\left[\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}a_{ls}v_{ws}x^sz^{w+1}\right].\end{aligned} \tag{22}$$

Next, considering the above equations, we obtain the following approximation:

$$\begin{aligned}D_x^{\beta}\left(\mathfrak{A}(x,t,z)z(x,t)\right) &= \sum_{r=1}^{s_1}\mathfrak{S}_r(t)D_x^{\beta}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}h_{ir}f_{jr}x^iz^{j+1}\right] \\ &= \sum_{r=1}^{s_1}\mathfrak{S}_r(t)\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}h_{ir}f_{jr}D_x^{\beta}\left(x^iz^{j+1}\right)\right],\end{aligned} \tag{23}$$

then, using the extra Caputo pseudo-operational matrix, we approximate the above relation. For example, for the special case, we have

$$D_x^{\beta}\left(x^iz(x,t)\right) \approx \Theta^T(x)\varphi^T(x,\beta,i)Z\hat{\Theta}(t).$$

Similarly, we obtain

$$\begin{aligned}D_x^{2\beta}\left(\mathfrak{B}(x,t,z)z(x,t)\right) &= \sum_{s=1}^{s_2}\mathfrak{V}_s(t)D_x^{2\beta}\left[\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}a_{ls}v_{ws}x^sz^{w+1}\right] \\ &= \sum_{s=1}^{s_2}\mathfrak{V}_s(t)\left[\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}a_{ls}v_{ws}D_x^{2\beta}\left(x^sz^{w+1}\right)\right],\end{aligned} \tag{24}$$

to derive $D_x^{2\beta}\left(x^sz^{w+1}\right)$, we can use the extra Caputo pseudo-operational matrix, too. As the next step, we put the all of approximations derived in this section, into Eqs. (1), (2) and the initial condition. Then, we get

$$\begin{aligned}\Lambda(x,t) &:= \Theta^T(x)Z\zeta(\alpha,t)\hat{\Theta}(t) \\ &- \left(-\sum_{r=1}^{s_1}\mathfrak{S}_r(t)\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}h_{ir}f_{jr}D_x^{\beta}\left(x^iz^{j+1}\right)\right]\right. \\ &+ \left.\sum_{s=1}^{s_2}\mathfrak{V}_s(t)\left[\sum_{l=0}^{\infty}\sum_{w=0}^{\infty}a_{ls}v_{ws}D_x^{2\beta}\left(x^sz^{w+1}\right)\right]\right),\end{aligned} \tag{25}$$

$$\begin{aligned}\Upsilon(x,t) &:= \Theta^T(x)Z\zeta(\alpha,t)\hat{\Theta}(t) \\ &- \left(-\mathfrak{A}(x,t)\left[\Theta^T(x)\zeta^T(\beta,x)Z\hat{\Theta}(t)\right]\right. \\ &+ \left.\mathfrak{B}(x,t)\left[\Theta^T(x)\zeta^T(2\beta,x)Z\hat{\Theta}(t)\right]\right),\end{aligned} \tag{26}$$

and

$$\Delta(x) := \Theta^T(x)Z\hat{\Theta}(0) - \mu(x). \tag{27}$$

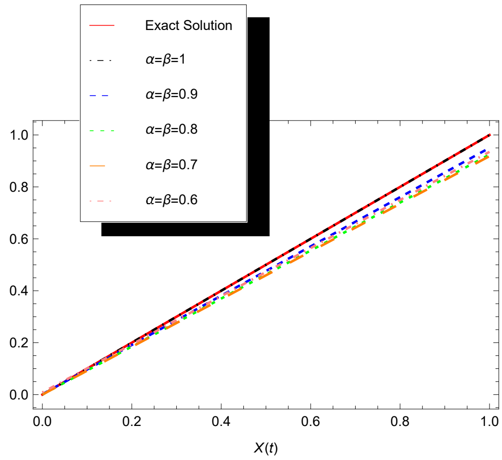

Figure 1. The exact and approximate solutions for $M = N = 2$ and different values of $\alpha = \beta$ in Example 1.

Table 1. Comparison of the numerical values of the present method and some existing methods for $\beta = 1$ and different values of α in Example 1.

x	t		$\alpha = 0.6$			$\alpha = 0.8$	
		TSM	CPFK	Our method	TSM	CPFK	Our method
0.2	0.25	0.6871	0.4497	0.4497	0.5542	0.4498	0.4498
	0.5	0.9384	0.6992	0.6992	0.8167	0.6996	0.6995
0.5	0.25	0.8871	0.6493	0.6492	0.7542	0.6496	0.6496
	0.5	1.1384	0.8995	0.8994	10.167	0.8997	0.8997

Now, in order to find the numerical solution of the mentioned equations, we apply the Lagrange multipliers method, design the following function

$$\mathfrak{J} \approx \mathfrak{J}^* = \mathfrak{J}[Z, \lambda] = \mathfrak{J}[Z] + \Pi\Omega, \tag{28}$$

where

$$\mathfrak{J}[Z] = \int_0^1 \int_0^1 \Lambda^2(x,t)\mathrm{dxdt}, \quad or \quad \mathfrak{J}[Z] = \int_0^1 \int_0^1 \Upsilon^2(x,t)\mathrm{dxdt}.$$

$$Pi = [\lambda_0, \lambda_1, \cdots, \lambda_{M-1}],$$

$$\Omega = [\omega_m]^T, \quad \omega_m = \Delta(x_m),$$

x_m, $m = 0, 1, \cdots, M-1$ are zeros of the shifted Legendre polynomials. Here, Ω is the vector containing the Lagrange multipliers. Therefore, in order to find an extremum for the problem, we solve the following system

$$\frac{\partial \mathfrak{J}^*}{\partial Z} = 0, \quad \frac{\partial \mathfrak{J}^*}{\partial \Omega} = 0.$$

Eventually, by computing Z, the numerical solution of the problems is approximately obtained in terms of the Touchard polynomials using Eq. (15).

5. Numerical results

Herein, we numerically study the proposed scheme on some test problems. We use Mathematica 12.0 for the numerical simulations.

Example 1. First, in Eq. (1), we consider $\mathfrak{A}(x,t,z) = -1, \mathfrak{B}(x,t,z) = 1$ and $\mu(x) = x$. The analytical solution of this problem for $\alpha = \beta = 1$ is $z(x,t) = x + t$. The scheme introduced in the previous section is utilized for solving this example with $M = N = 2$. For $\alpha = \beta = 1$, we derive the exact solution of the example, while the absolute error of Cubic B-Spline scaling function method for $M = 3$ are reported in Ref. [10]. It is clear that the present scheme is more accurate than expressed method. Fig. 1 shows the approximate solutions attained for the example at different values of $\alpha = \beta$. Besides, in Table 1, we reported the numerical values obtained by the present method ($M = N = 2$), Taylor series method (TSM) [7], and Chebyshev polynomial of the fourth kind (CPFK) [6].

Example 2. In Eq. (1), we consider $\mathfrak{A}(x,t,z) = x, \mathfrak{B}(x,t,z) = \frac{x^2}{2}$ and $\mu(x) = x$. The analytical solution of this problem for $\alpha = \beta = 1$ is $z(x,t) = xe^t$. In Table 2, we report the comparison of the absolute errors of the present method and finite

Table 2. Comparison of the absolute errors of the present method and finite difference method for $\alpha = \beta = 1$ in Example 2.

x	finite difference method $h = 0.1$	finite difference method $h = 0.0125$	Present method $M = 2, N = 8$
0.2	$3.6010 \times E^{-10}$	$1.3576 \times E^{-12}$	$2.59785 \times E^{-12}$
0.4	$5.0079 \times E^{-09}$	$1.9580 \times E^{-11}$	$2.59646 \times E^{-12}$
0.6	$1.2281 \times E^{-08}$	$4.7970 \times E^{-11}$	$2.59507 \times E^{-12}$
0.8	$2.0546 \times E^{-08}$	$8.0241 \times E^{-11}$	$2.59357 \times E^{-12}$

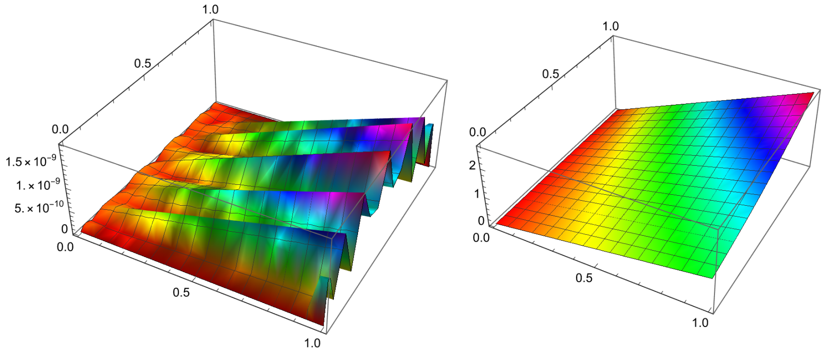

Figure 2. The plot of absolute error (left) and the approximate solution (right) for $M = 2, N = 8$ and $\alpha = \beta = 1$ in Example 2.

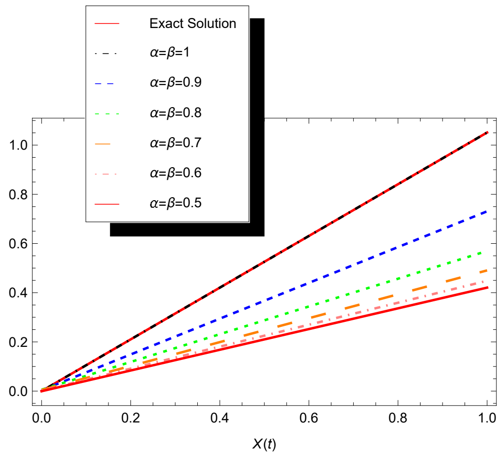

Figure 3. The exact and approximate solutions for $M = 2, N = 6$ and different values of $\alpha = \beta$ in Example 2.

difference method [21] at $t = 1$. Also, to show the accuracy of the proposed technique, the plot of absolute error and the approximate solution for $M = 2, N = 8$ and $\alpha = \beta = 1$ is displayed in Fig. 2. Also, Fig. 3 shows the approximate solutions attained for the example at different values of $\alpha = \beta$ and $M = 2, N = 6$. For more investigation, for $\alpha, \beta \neq 1$, we measure the reliability by defining the following residual error function

$$R(x,t) = \left| D_t^{\alpha} z(x,t) - \left(-D_x^{\beta} x + D_x^{2\beta} \frac{x^2}{2} \right) z(x,t) \right|,$$

Fig. 4 displays the plot of the residual error function for different values of α, β and $M = 2, N = 8$.

Example 3. In Eq. (1), we consider $\mathfrak{A}(x,t,z) = \frac{7}{2} z, \mathfrak{B}(x,t,z) = xz$ and $\mu(x) = x$. The analytical solution of this problem for $\alpha = \beta = 1$ is $z(x,t) = \frac{x}{1+t}$. We solve this problem by applying the suggested method. Fig. 5 demonstrate the approximate solutions attained for the example at different values of $\alpha = \beta$ and $M = 2, N = 5$. In addition, the plot of absolute error and the approximate solution for $\alpha = \beta = 1$, $M = 2, N = 4$, and $N = 6$ are shown in Fig. 6.

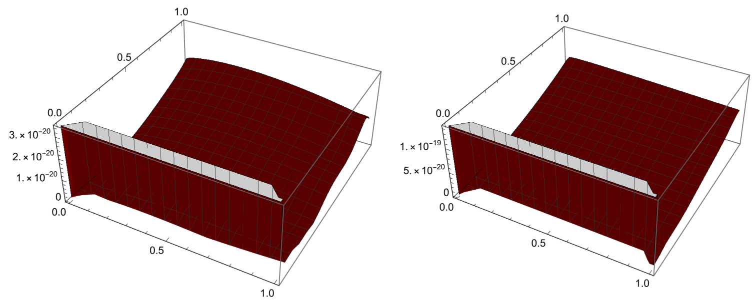

Figure 4. The plot of the residual error function for $\alpha = 0.99, \beta = 0.9$ (left) $\alpha = 0.9, \beta = 0.7$ (right) and $M = 2, N = 8$ in Example 2.

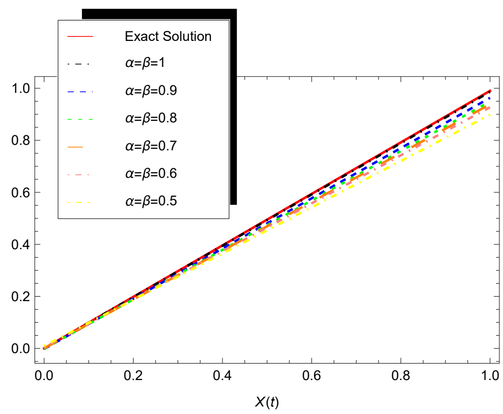

Figure 5. The exact and approximate solutions for $M = 2, N = 5$ and different values of $\alpha = \beta$ in Example 3.

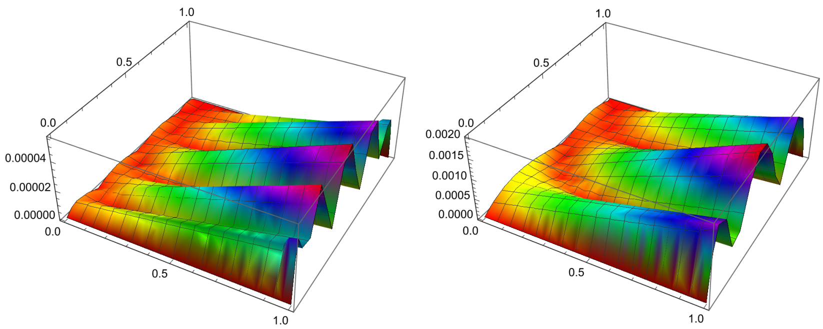

Figure 6. The plot of absolute error for $\alpha = \beta = 1$, $M = 2, N = 6$ (left) and $M = 2, N = 4$ (right) in Example 3.

6. Conclusions

In this article, we proposed an efficient numerical method to solve fractional-order Fokker-Planck equations. To design this method, we used Touchard polynomials. First, we presented Caputo and extra Caputo pseudo-operational matrices for these polynomials. With the help of an optimal method, the considered equation led to a system of algebraic equations. At last, illustrative examples are given to display the validity and effectiveness of the developed technique. Probably, this method to solve equations with many conditions is not suitable, because in this case, CPU time used is increased over some approximation methods. In addition, stability analysis of the suggested method for the numerical solution of the mentioned problem and applying the method for solving various kinds of fractional equations such as fractional optimal control problems, and delay fractional differential equations are interesting problems for future works.

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