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Newton-Krylov generalized minimal residual algorithm in solving the nonlinear two-dimensional integral equations of the second kind on non-rectangular domains with an error estimate

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In this paper, an applicable numerical approximation has been proposed for solving nonlinear two-dimensional integral equations (2DIEs) of the second kind on non-rectangular domains. Because directly applying the collocation methods on non-rectangular domains is difficult, in this work, at first, the integral equation is converted to an equal integral equation on a rectangular domain, then the solution is approximated by applying 2D Jacobi collocation method, the implementation of these instructions reduces the integral equation to a system of nonlinear algebraic equations, therefore, solving this system has an important role to approximate the solution. In this paper, Newton-Krylov generalized minimal residual (NK-GMRes) algorithm is used for solving the system of nonlinear algebraic equations. Furthermore, an error estimate for the presented method is investigated and several examples confirm the accuracy and efficiency of the proposed instructions.

Keywords: Non-rectangular domains; 2D integral equations; Jacobi polynomials; Collocation method; Newton-Krylov GMRes algorithm.

1. Introduction

Integral equations occur in wide variety of problems and provide an important tool for modeling the numerous problems in physics, engineering, mechanics, and applied sciences. In [1], a 2DIE was investigated that appear in axisymmetric contact problems for bodies with complex rheology. The formulation of a problem in diffraction theory leads us to solve a 2D singular integral equation [2], in [3], a 2D Fredholm integral equation of the first kind was discussed that this equation has been raised in potential problems for elliptic and circular disk, an integral equation appears in connection with the problem of the electrostatic potential due to a charged disk [5], the investigation of mixed problems of the mechanics of continuous media leads us to solve an integral equation [6], see [4, 7, 8, 9, 10, 11]. Due to the limited scope of analytical solutions, attentions have been paid for approximating the solution of these equations. There are various numerical methods for solving integral equations. In [12] Taylor polynomial solutions were proposed to solve non-linear Volterra-Fredholm (V-F) integral equations. In [13], a Taylor method was improved and implemented on high-order linear V-F integro-differential equations. Shahmorad in [14] solved general form linear Fredholm-Volterra integro-differential equations by the Tau method and also error estimate of the method was investigated. In [15], mixed V-F integral equations by applying 2D Bernstein polynomials and their operational matrix have been approximated. In [16], rationalizing Haar wavelet functions has been applied to solve 2DIEs. In [17], Jacobi collocation method was investigated for solving multi-dimensional Volterra integral equations. Mirzaee et al. In [18], discussed three-dimensional V-F integral equations, three-dimensional V-F integral equations of the first and second kinds were approximated using Bernstein's approximation in

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[19], for more references and other methods in applied sciences see [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].
Notice that all the reviewed nonlinear 2DIEs are on rectangular domains, here we would like to consider a larger class of nonlinear 2DIEs on non-rectangular domains as follows:

$$u(x,t) - \sum_{j=1}^{r} \lambda_j \int_D k_j(x,t,y,s,u(y,s))dyds = f(x,t), \quad (x,t) \in D, \tag{1}$$

where $u(x,t)$ is an unknown function, k_j, f are the known functions, r is a positive integer constant, λ_j are real constants, and D is the non-rectangular domain of the equation.
The domain of this kind of integral equations merely is not rectangular, therefore, directly applying the collocation method for solving them is difficult. For solving 2D integral equations on non-rectangular domains using collocation, Galerkin and Nystrom methods, an instruction has been applied in [31], mesh refinement and triangulation increase the difficulties and complexity of the problems. In [32, 33, 34, 35] have been proposed other instructions.
Collocation method is a popular, powerful, and simple method that has been considered allots in past decades, for example see [36, 37, 38, 39, 40]. In this work, at first, the domain of the problems is transferred to a rectangular domain and then 2D shifted Jacobi collocation is applied for solving them. By utilizing these instructions, the problem is reduced to a system of nonlinear algebraic equations. So speed and accuracy of the method to solve this system are important parameters to approximate this type of integral equations. Newton-Krylov algorithm is one of the good ideas and most powerful that has been considered by the amount large of researchers, see [43, 44, 45].
The rest of the paper is as: in Section 1.1, some properties of Jacobi and shifted Jacobi polynomials are discussed. In Section 1.2, the NK-GMRes algorithm is presented, in Section 2, the implementation of the method is explained. Section 3 is devoted to convergence analysis. In Section 4, the numerical results are reported, and in the last section, a conclusion of our study is provided.

1.1. Jacobi and shifted Jacobi polynomials

In this section, some fundamental concepts and properties involved with orthogonal shifted Jacobi polynomials are recalled. The Jacobi polynomials, associated with the real parameters $\alpha, \beta > -1$, are a sequence of polynomials $J_i^{(\alpha,\beta)}(x)(i = 0, 1, ...)$ of degree i and orthogonal with respect to the weight function $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$ on the interval $[-1, 1]$:

$$\int_{-1}^{1} J_m^{(\alpha,\beta)}(x) J_n^{(\alpha,\beta)}(x) w(x) = h_m \delta_{mn},$$

where δ_{mn} is the Kronecker function and

$$h_m = \frac{2^{(\alpha+\beta+1)}\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(2m+\alpha+\beta+1)m!\Gamma(m+\alpha+\beta+1)}.$$

The shifted Jacobi polynomials of $P_k^{(\alpha,\beta)}(x)$ are obtained by applying the mapping of $x \to (2x-1)$ to the Jacobi polynomials [46]. The analytical form of them is given by

$$P_k^{(\alpha,\beta)}(x) = \sum_{i=0}^{k} \frac{(-1)^{(k-i)}\Gamma(k+\beta+1)\Gamma(k+i+\alpha+\beta+1)}{\Gamma(i+\beta+1)\Gamma(k+\alpha+\beta+1)(k-i)!i!} x^i,$$

and also they are orthogonal on $[0, 1]$ as:

$$\int_0^1 P_i^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(x) w^{(\alpha,\beta)}(x) dx = \eta_k \delta_{ik},$$

where $w^{(\alpha,\beta)}(x) = x^{\beta}(1-x)^{\alpha}$ and $\eta_k = \frac{h_k}{2^{(\alpha+\beta+1)}}$. More properties of these functions are stated in Ref. [46].
Any arbitrary function of $u(x) \in L^2([0, 1])$ can be expanded as:

$$u(x) = \sum_{i=0}^{\infty} a_i P_i^{(\alpha,\beta)}(x), \tag{2}$$

where

$$a_i = \frac{1}{\eta_i} \int_0^1 P_i^{(\alpha,\beta)}(x) u(x) w^{(\alpha,\beta)}(x) dx, \quad i = 0, 1, \cdots.$$

If the infinite series in Eq. (2) is truncated, then it can be written as

$$u(x) \simeq u_N(x) = \sum_{i=0}^{N} a_i P_i^{(\alpha,\beta)}(x) = P^T(x)A,$$

where $A = [a_0, a_1, ..., a_N]^T$ and $P(x) = [P_0^{(\alpha,\beta)}(x), P_1^{(\alpha,\beta)}(x), ..., P_N^{(\alpha,\beta)}(x)]^T$.
Two variables shifted Jacobi polynomials are defined as:

$$T_{mn}^{(\alpha,\beta)}(x,y) = P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y), \quad (x,y) \in D^2 = [0,1]\times[0,1], \text{ and } \quad m,n = 0,1,\cdots.$$

These polynomials with the weight function of $W^{(\alpha,\beta)}(x,y) = w^{(\alpha,\beta)}(x)w^{(\alpha,\beta)}(y)$ on D^2 are orthogonal:
Lemma 1.1. Two variables shifted Jacobi polynomials of $T_{mn}^{(\alpha,\beta)}(x,y)$ are orthogonal on D^2.
Proof. See Ref. [46].

A function $u(x,y)$ defined over D^2 can be expressed in terms of two variables shifted Jacobi polynomials as

$$u(x,y) = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij}T_{ij}^{(\alpha,\beta)}(x,y), \quad (x,y) \in D^2,$$

where

$$a_{ij} = \frac{1}{\eta_i\eta_j}\int_0^1\int_0^1 T_{ij}^{(\alpha,\beta)}(x,y)u(x,y)W^{(\alpha,\beta)}(x,y)dxdy.$$

If we approximate $u(x,y)$ by the first $(N+1)(M+1)$-terms, then we can write

$$u(x,y) \simeq u_{NM}(x,y) = \sum_{j=0}^{M}\sum_{i=0}^{N} a_{ij}T_{ij}^{(\alpha,\beta)}(x,y) = \Phi^T(x,y)A,$$

where $A = [a_{00},\dots,a_{0M},\dots,a_{N0},\dots,a_{NM}]^T$ and

$$\Phi(x,y) = [T_{00}^{(\alpha,\beta)}(x,y),\dots,T_{0M}^{(\alpha,\beta)}(x,y),\dots,T_{N0}^{(\alpha,\beta)}(x,y),\dots,T_{NM}^{(\alpha,\beta)}(x,y)]^T.$$

1.2. Newton-Krylov algorithm

By applying spectral methods on nonlinear integral equations, a system of nonlinear algebraic equations is obtained. Let $F(x)=0$ where $F:\mathbb{R}^n \to \mathbb{R}^n$ is a function of $F(x) = (f_1(x), f_2(x),\dots,f_n(x))^T$ and $x \in \mathbb{R}^n$ is a vector. Consider the well-known Newton's iterative method:

$$F(x_{n+1}) = F(x_n) + (x_n - x_{n+1})F'(x_n),$$

$F'(x_n)$ is a $n\times n$ Jacobian matrix. Therefore

$$x_{n+1} = x_n - J(x_n)^{-1}F(x_n).$$

It is obvious at each iteration, a linear system must be solved, by increasing the complexity of equations solving this linear system mostly could be time-consuming process. One of the best ideas to overcome this problem is to use from the Newton-Krylov generalized minimal residual algorithm (NK-GMRes):

A framework for GMRes implementation

Begin
1. $r = b - Ax,\ v_1 = r/||r||_2,\ \rho = ||r||_2,\ \beta = \rho,\ k = 0.$
2. *while* $\rho > \epsilon||b||_2$ and $k < k_{max}$ *do*
   - **2.a** $k = k+1.$
   - **2.b** Apply Arnoldi to obtain H_k and V_{k+1} from V_k and H_{k-1}.
   - **2.c** $e_1 = (1,0,\dots,0)^T \in R^{k+1}$.
   - **2.d** Solve min $||\beta e_1 - H_k y_k||_{R^{k+1}}$ for $y_k \in R^k$.
   - **2.e** $\rho = ||\beta e_1 - H_k y_k||_{R^{k+1}}$.

   end while
3. $x_k = x_0 + V_k y_k$.

End

For more details see [47].

2. Proposed Method

At first, the method is applied to a particular non-rectangular domain that we call it the domain of the first kind, then we will show this method can be extended for solving other kinds of 2D non-rectangular domains.

2.1. The domain of the first kind:

If

$$D = \{(\theta_1, \theta_2) \in \mathbb{R}^2 : a \le \theta_1 \le b, \alpha(\theta_1) \le \theta_2 \le \beta(\theta_1)\},$$

we will have

$$u(\theta_1, \theta_2) - \sum_{j=1}^{r} \lambda_j \int_a^b \int_{\alpha(\rho)}^{\beta(\rho)} k_j(\theta_1, \theta_2, \rho, \tau, u(\rho, \tau)) d\tau d\rho = f(\theta_1, \theta_2), \quad (3)$$

where $a \le \theta_1 \le b$ and $\alpha(\theta_1) \le \theta_2 \le \beta(\theta_1)$. θ_1 and θ_2 are converted to the fixed interval $[0, 1]$ by linear transformation as follows

$$\theta_2 = y\beta(\theta_1) + (1 - y)\alpha(\theta_1), \quad \theta_1 = (b - a)x + a, \quad (4)$$

where $x, y \in [0, 1]$. By substitution right-hand sides of the above relations in $u(\theta_1, \theta_2)$, we have

$$u(\theta_1, \theta_2) = u\Big((b - a)x + a, y\beta(\theta_1) + (1 - y)\alpha(\theta_1)\Big) = u^*(x, y).$$

Now, the domain of $u^*(x, y)$ is squared, and as mentioned in the past section, may be approximated as

$$u^*(x, y) \simeq u^*_{NM}(x, y) = \sum_{j=0}^{M} \sum_{i=0}^{N} a_{ij} T_{ij}^{(\alpha,\beta)}(x, y),$$

since $u(\theta_1, \theta_2) = u^*(x, y)$, $u^*_{NM}(x, y)$ is also an approximation of $u(\theta_1, \theta_2)$. In another word, we will have

$$u_{NM}(\theta_1, \theta_2) = u^*_{NM}(x, y) \simeq u^*(x, y) = u(\theta_1, \theta_2).$$

By substitution $u^*_{NM}(x, y)$ in Eq. (3), we have:

$$u_{NM}(x, y) - \sum_{j=1}^{r} \lambda_j \int_a^b \int_{\alpha(\rho)}^{\beta(\rho)} k_j(\theta_1, \theta_2, \rho, \tau, u_{NM}(\rho, \tau)) d\tau d\rho = f(\theta_1, \theta_2). \quad (5)$$

Utilizing Gauss-Legendre integration rule [41] and using $\{(x_i, y_j), i = 0 \dots N \text{ and } j = 0 \dots M\}$ as collocation points, where $\{x_i\}_{i=0}^N$ and $\{y_j\}_{j=0}^M$ are zeros of the shifted Jacobi polynomials of degree $(N + 1)$ and $(M + 1)$, respectively, that by using expressions in Eq. (4) are transferred to the domain of the problem, $(N + 1)(M + 1)$ nonlinear algebraic equations are generated that to solve this nonlinear system of algebraic equations and obtaining unknown coefficients we have used the NK-GMRes method.

2.2. The domain of the second kind:

If

$$D = \{(\theta_1, \theta_2) \in \mathbb{R}^2 : \beta(\theta_2) \le \theta_1 \le \alpha(\theta_2), a \le \theta_2 \le b\}.$$

Similarly, by commuting the variables, it is clear that the computations are straightforward.

Remark. If a domain is not either the first or second kinds but could be divided into the finite number of the first and second kinds sub-domains, then the method is applied in each sub-domains. In another word, suppose

$$D = \{D_1 \bigcup D_2 \bigcup \dots \bigcup D_q\}, \quad q \in \mathbb{N},$$

where D_i belongs to the domain of the first or second kind, by applying the method on domain i, we have

$$u^i_{NM}(\theta_1, \theta_2) - \sum_{j=1}^{r} \lambda_j \int_D k_j(\theta_1, \theta_2, \rho, \tau, u^i_{NM}(\rho, \tau)) d\tau d\rho = f(\theta_1, \theta_2), \quad (6)$$

where $u^i_{NM}(\theta_1, \theta_2)$ is the approximation function on domain D_i. Now, there exist $(N + 1)(M + 1)$ unknown coefficients, by using $[\frac{(N+1)(M+1)}{q}]$ number of scattered points in each sub-domain as collocation points, the problem is converted to find the solution for the system of nonlinear algebraic equations.

3. Error estimate

In this section, some theorems related to convergence analysis and error estimate are presented that the following theorems show by increasing N and M, the approximate solution $u_{NM}(\theta_1,\theta_2)$ is convergent to $u(\theta_1,\theta_2)$.

Theorem 3.1. Let $u(x,y)\in C^M[0,1]\times C^M[0,1]$ and is approximated such as $u_{MM}(x,y)=\sum_{j=0}^{M}\sum_{i=0}^{M}a_{ij}T_{ij}^{(\alpha,\beta)}(x,y)$, then we have

$$|a_{ij}|\leq\frac{1}{2^{2(i+j)}}L_{ij}^{(\alpha,\beta)}max_{x\in D}\left|\frac{\partial^{(i+j)}u(x,y)}{\partial x^i\partial y^j}\right|,\qquad D=[0,1]\times[0,1],$$

where $L_{ij}^{(\alpha,\beta)}$ is independent of the function of $u(x,y)$.
Proof. See [42].

Theorem 3.2. Suppose $u(x,y)\in C^M[0,1]\times C^M[0,1]$ then the bound of the error for the approximate solution resulted is as follows:

$$\|u(x,y)-\Phi^T(x,y)A\|\leq\frac{2^{(2M)}}{(2M)!}E,$$

where $E=max\{E_0,E_1,...,E_{2M}\}$, and

$$E_i=max_{(x,y)\in D}\|\frac{\partial^{(2M)}u(x,y)}{\partial x^{(2M-i)}\partial y^i}\|,\quad i=0,...,2M.$$

Proof. Let $Z_M(x,y)$ of degree at most M with respect to both variables x and y which interpolated $u(x,y)$ in the domain D, therefore we will have

$$\int_0^1\int_0^1 W^{(\alpha,\beta)}(x,y)\Big(u(x,y)-\Phi^T(x,y)A\Big)^2dxdy\leq\int_0^1\int_0^1 W^{(\alpha,\beta)}(x,y)\Big(u(x,y)-Z_M(x,y)\Big)^2dxdy,$$

then consider the Taylor expansion about $(0,0)$ in D, the bound of error is obtained as follows:

$$u(x,y)-Z_M(x,y)=\sum_{i=0}^{2M}\frac{\partial^{(2M)}u(\xi_x,\xi_y)}{(2M-i)!i!\partial x^{(2M-i)}\partial y^i}x^{(2M-i)}y^i,$$

that $(\xi_x,\xi_y)\in[0,1]\times[0,1]$. Therefore

$$\|u(x,y)-Z_M(x,y)\|\leq\sum_{i=0}^{2M}\frac{1}{(2M-i)!i!}max_{(\xi_x,\xi_y)\in[0,x]\times[0,y]}\left\|\frac{\partial^{(2M)}u(\xi_x,\xi_y)}{\partial x^{(2M-i)}\partial y^i}x^{(2M-i)}y^i\right\|.$$

Since

$$max_{(\xi_x,\xi_y)\in[0,x]\times[0,y]}\left\|\frac{\partial^{(2M)}u(\xi_x,\xi_y)}{\partial x^{(2M-i)}\partial y^i}\right\|\leq max_{(x,y)\in D}\left\|\frac{\partial^{(2M)}u(x,y)}{\partial x^{(2M-i)}\partial y^i}\right\|=N_i.$$

Therefore

$$\|u(x,y)-Z_M(x,y)\|\leq\sum_{i=0}^{2M}\frac{N_i}{(2M-i)!i!}.$$

Consider $N=\max\,[N_0,N_1,...,N_{2M}]$, hence we have

$$\|u(x,y)-Z_M(x,y)\|\leq\sum_{i=0}^{2M}\frac{N}{(2M-i)!i!}=\frac{N}{2M!}\sum_{i=0}^{2M}\binom{2M}{i}=\frac{N2^{(2M)}}{2M!},$$

therefore

$$\|u(x,y)-\Phi^T(x,y)A\|\leq\frac{N2^{(2M)}}{2M!}.\quad\square$$

4. Illustrative Examples

In order to illustrate the performance of the presented method in solving 2D integral equations on the non-rectangular domain and justify the efficiency of the method, the following examples have been considered. To study the convergence behavior of the method, the maximum and mean errors have been used with the following definition, respectively:

$$\|e_{(N,M)}\|_\infty=max_{(x,t)\in D}\Big|u_{ex}(x,t)-u_{NM}(x,t)\Big|$$

$$\|e_{(N,M)}\|_2 = \left(\int_D \left(u_{ex}(x,t) - u_{NM}(x,t)\right)^2 dxdt \right)^{\frac{1}{2}}.$$

where $u_{ex}(x,t)$ and $u_{NM}(x,t)$ are the exact and approximate solutions.
The numerical implementation is carried out in the Microsoft Maple 15 with hardware configuration: Desktop 32-bit Intel Core 2 Due CPU, 4 GB of RAM, 32-bit operation system.

Example 4.1. Consider the nonlinear Fredholm integral equation is given in [34] by

$$u(x,t) - \int_D (\frac{x}{1+t})(1+y+s)u^2(y,s)dyds = \frac{1}{(x+t+1)^2} - \frac{0.0271380471x}{1+t}, \quad (x,t) \in D, \tag{7}$$

where D is drowned in Figure 1(a) and determined by

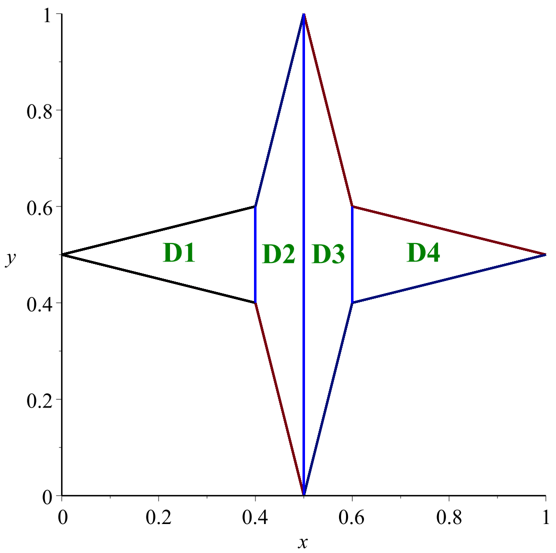
$$D_1 = \left\{(x,t) \in \mathbb{R}^2: \quad 0 \le x \le \tfrac{4}{10}, \quad (-\tfrac{1}{4})x + \tfrac{1}{2} \le t \le (\tfrac{1}{4})x + \tfrac{1}{2}\right\},$$

$$D_2 = \left\{(x,t) \in \mathbb{R}^2: \quad \tfrac{4}{10} \le x \le \tfrac{1}{2}, \quad -4x + 2 \le t \le 4x - 1\right\},$$

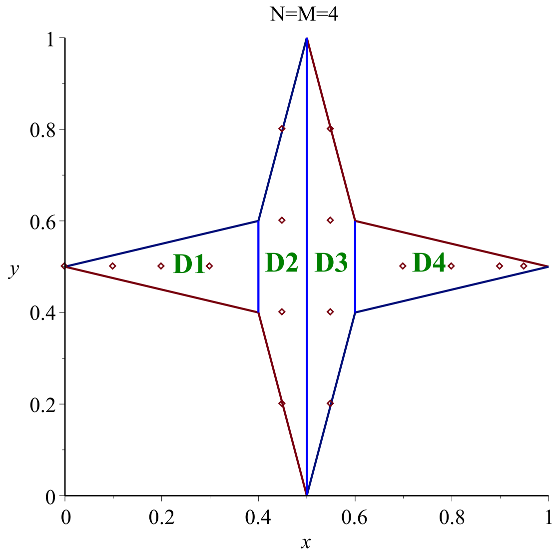
$$D_3 = \left\{(x,t) \in \mathbb{R}^2: \quad \tfrac{1}{2} \le x \le \tfrac{6}{10}, \quad 4x - 2 \le t \le -4x + 3\right\},$$

$$D_4 = \left\{(x,t) \in \mathbb{R}^2: \quad \tfrac{6}{10} \le x \le 1, \quad (\tfrac{1}{4})x + \tfrac{1}{4} \le t \le (\tfrac{-1}{4})x + \tfrac{3}{4}\right\},$$

and has the exact solution $u(x,t) = \frac{1}{(x+t+1)^2}$. We have applied the mentioned method and solved Eq (7). The distribution of collocation points is depicted in Figure 1(b) and also the numerical results of $\|e\|_2$, $\|e\|_\infty$ and the rate of convergence at different points of N and M are presented in Table 1. The obtained results are compared with the method in [34]. Figure 2 shows the absolute error for $N = M = 5$.



(a) The consideration domain D for example 4.1.



(b) Nodes distribution of example 4.1.

Figure 1. The consideration domain D and the nodes distribution for example 4.1

Example 4.2. Consider the nonlinear Fredholm integral equation is given in [34] by

$$u(x,t) - \int_D \frac{x(1-y^2)}{(1+t)(1+s^2)}(1 - e^{-u(y,s)})dsdy = -ln(1 + \frac{xt}{1+t^2}) + \frac{0.0321559973x}{1+t}, \quad (x,t) \in D, \tag{8}$$

where D is drowned in Figure 3(a) and determined by

$$D_1 = \left\{(x,t) \in \mathbb{R}^2: \quad 0 \le x \le \tfrac{1}{6}, \quad 1 - 6x \le t \le 1\right\},$$

Table 1. Some numerical results for example 4.1.

Present Method					**Method in [34]**		
N = M	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Ratio	Iteration	N	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$
2	1.35×10^{-2}	4.07×10^{-2}	-	7	10	4.39×10^{-3}	2.71×10^{-2}
3	4.41×10^{-3}	9.12×10^{-3}	4	11	18	2.21×10^{-3}	1.36×10^{-2}
4	1.24×10^{-3}	4.56×10^{-3}	2	11	25	9.21×10^{-4}	8.53×10^{-3}
5	9.74×10^{-5}	2.55×10^{-4}	17	11	38	2.37×10^{-5}	6.23×10^{-3}

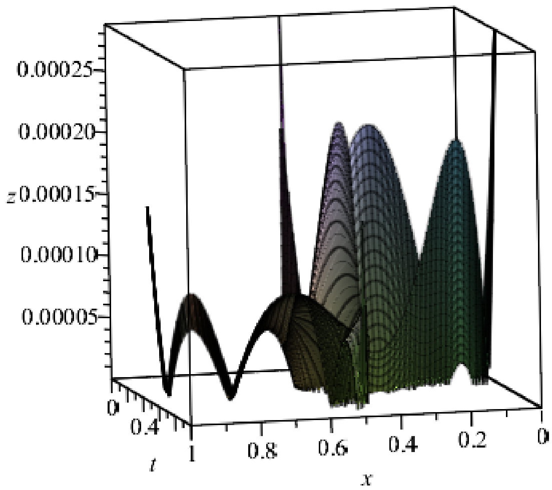


Figure 2. Absolute error of example 4.1 for N=M=5.

$$D_2 = \left\{(x,t) \in \mathbb{R}^2 : \ \tfrac{1}{6} \leq x \leq \tfrac{2}{7}, \quad 0 \leq t \leq (\tfrac{191}{100}) - \tfrac{273x}{50}\right\},$$
$$D_3 = \left\{(x,t) \in \mathbb{R}^2 : \ \tfrac{2}{7} \leq x \leq \tfrac{1}{2}, \quad (-\tfrac{4}{5}) + (\tfrac{14}{5})x \leq t \leq (\tfrac{91}{30})x + \tfrac{-31}{60}\right\},$$
$$D_4 = \left\{(x,t) \in \mathbb{R}^2 : \ \tfrac{1}{2} \leq x \leq \tfrac{5}{7}, \quad (-\tfrac{1}{4})x + \tfrac{1}{2} \leq t \leq (\tfrac{1}{4})x + \tfrac{1}{2}\right\},$$
$$D_5 = \left\{(x,t) \in \mathbb{R}^2 : \ \tfrac{5}{7} \leq x \leq \tfrac{5}{6}, \quad 0 \leq t \leq (\tfrac{273}{50})x - \tfrac{71}{20}\right\},$$
$$D_6 = \left\{(x,t) \in \mathbb{R}^2 : \ \tfrac{5}{6} \leq x \leq 1, \quad -5 + 6x \leq t \leq 1\right\},$$

and has the exact solution $u(x,t) = -ln(1 + \frac{xt}{1+t^2})$. The distribution of nodes is depicted in Figure 3(b), and also Table 2 presents the numerical results of $\|e\|_2$, $\|e\|_\infty$ and the rate of convergence at different points of N and M. The obtained results are compared with the method in [34]. Figure 4 shows the absolute error for $N = M = 6$.

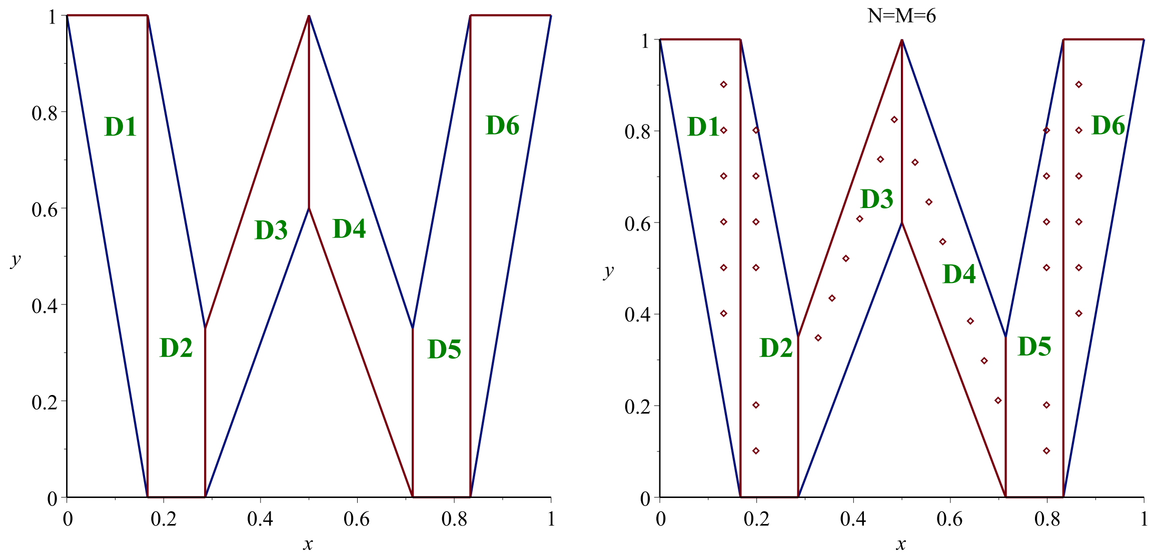
Table 2. Some numerical results for example 4.2.

Present Method					**Method in [34]**		
N = M	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Ratio	Iteration	N	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$
2	2.11×10^{-1}	4.49×10^{-1}	-	1	10	1.55×10^{-3}	9.86×10^{-3}
4	1.74×10^{-3}	3.11×10^{-2}	14	2	18	1.73×10^{-4}	6.77×10^{-4}
5	4.16×10^{-4}	1.04×10^{-3}	29	2	25	6.22×10^{-5}	2.14×10^{-4}
6	4.89×10^{-5}	1.64×10^{-4}	6	1	38	9.24×10^{-5}	2.26×10^{-4}

Example 4.3. Consider the nonlinear Fredholm integral equation with the exact solution $u(x,t) = t\sin(x)$:

$$u(x,t) - \int_D \left(u(y,s)\right)^2 ds dy = t\sin(x) - 0.02800777433, \quad (x,t) \in D, \tag{9}$$

where is shown in Figure 5(a) and determined by



(a) The consideration domain D for example 4.2. (b) Nodes distribution for example 4.2.

Figure 3. The consideration domain D and the nodes distribution for example 4.2.

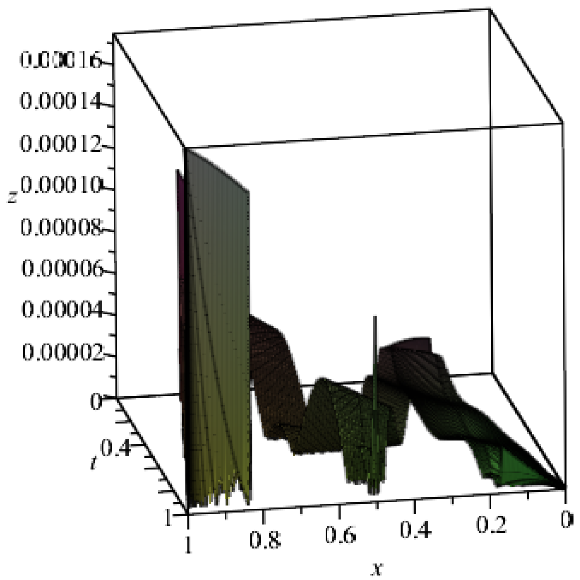


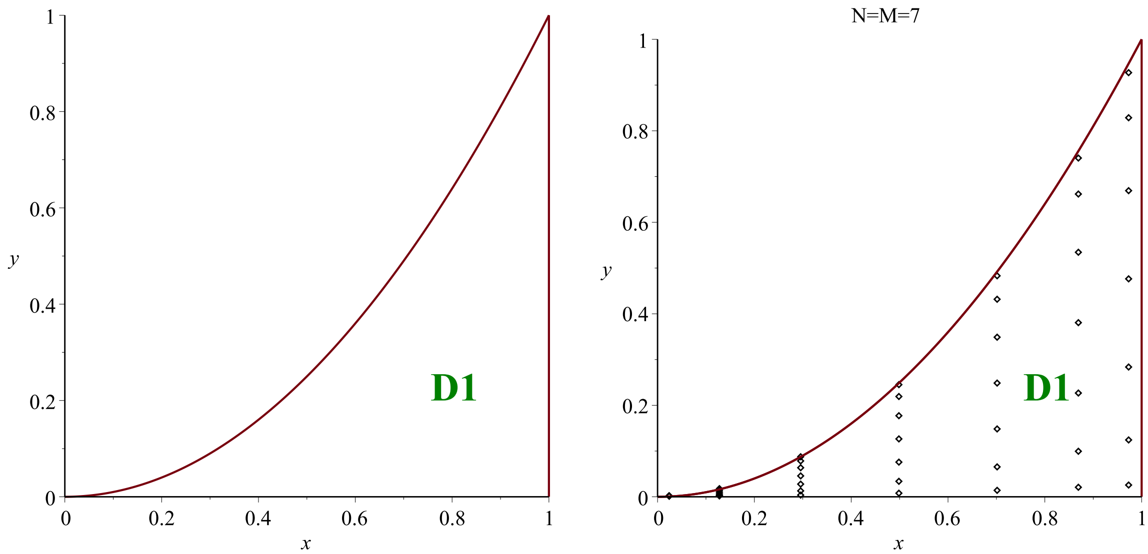
Figure 4. Absolute error of example 4.2 for N=M=6.

$$D = \left\{(x, t) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq t \leq x^2\right\}.$$

Figure 5(b) shows the distribution of collocation points. Table 3 presents the numerical results of $\|e\|_2$, $\|e\|_\infty$ and the rate of convergence at the different points of N and M. Then in Figure 6 the absolute error for $N = M = 7$ is presented.

5. Conclusion

In this study, 2D collocation method is developed for solving nonlinear 2DIEs on non-rectangular domains. Since applying of the 2D collocation method for non-rectangular is difficult, the domain is transferred onto a rectangular domain, and by using this simple trick could get relieved of this problem. By applying these instructions the problems are reduced to a system of nonlinear



(a) The consideration domain D for example 4.3.

(b) Nodes distribution for example 4.3.

Figure 5. The consideration domain D and the nodes distribution for example 4.3.

Table 3. some numerical results for example 4.3.

N	$\|\|e\|\|_2$	$\|\|e\|\|_\infty$	Ratio	Iteration
2	1.79×10^{-1}	3.65×10^{-1}	-	2
4	4.38×10^{-3}	8.57×10^{-3}	42	3
5	6.49×10^{-4}	1.87×10^{-3}	4	4
6	9.83×10^{-5}	8.95×10^{-4}	14	7
7	2.18×10^{-5}	1.29×10^{-4}	6	7

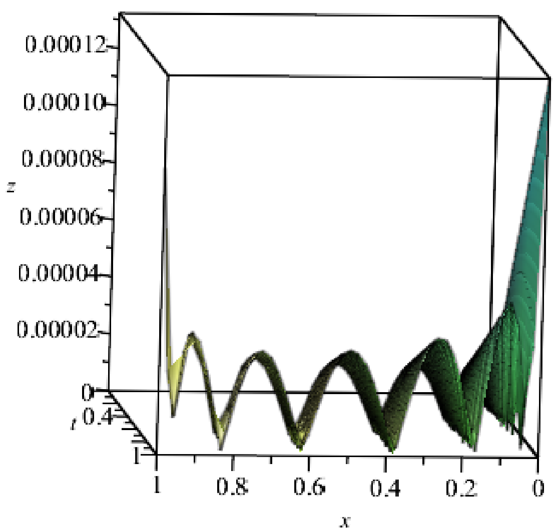


Figure 6. Absolute error of example 4.3 for N=N=7.

algebraic equations, therefore, running time and accuracy in solving this system are important parameters to approximate the solution of nonlinear 2DIEs on non-rectangular domains. In this work, the NK-GMRes method was used for solving the obtained system of nonlinear algebraic equations. Additionally, the convergence of accuracy is examined in three various nonlinear 2D integral equations, that confirm the accuracy and efficiency of the method.

References

1. A.V. Manzhirov. On a method of solving two-dimensional integral equations of axisymmetric contact problems for bodies with complex rheology. *Journal of Applied Mathematics and Mechanics*, 49(6):777-782, 1985.
2. J. Radlow. A two-dimensional singular integral equation of diffraction theory. *Bulletin of the American Mathematical Society*, 70(4):596-599, 1964
3. J. Boersma and E. Danick. On the solution of an integral equation arising in potential problems for circular and elliptic disks. *SIAM Journal on Applied Mathematics*, 53(4):931-941, 1993.
4. M.V. Mirkin and A.J. Bard. Multidimensional integral equations: a new approach to solving microelectrode diffusion problems: part 2. Applications to microband electrodes and the scanning electrochemical microscope. *Journal of Electroanalytical Chemistry*, 323(1-2):29-51,1992.
5. P. Wolfe. Eigenfunctions of the integral equation for the potential of the charged disk. *Journal of Mathematical Physics*, 12(7):1215-1218, 1971.
6. E.V. Kovalenko. Some approximate methods of solving integral equations of mixed problems. *Journal of Applied Mathematics and Mechanics*, 53(1): 85-92, 1989.
7. S. Hatamzadeh-Varmazyar, M. Naser-Moghadasi, E. Babolian, and Z. Masouri. Numerical approach to survey the problem of electromagnetic scattering from resistive strips based on using a set of orthogonal basis functions. *Progress In Electromagnetics Research*, 81:393-412, 2008.
8. M. S. Tong. A stable integral equation solver for electromagnetic scattering by large scatterers with concave surface. *Progress In Electromagnetics Research*, 74:113-130, 2007.
9. R. Cont and E.Voltchkova.Integro-differential equations for option prices in exponential Levy models. *Finance and Stochastics*, 9(3):299-325, 2005.
10. R. Ansari, K. Hosseini, A. Darvizeh and B. Daneshian. A sixth-order compact finite difference method for non-classical vibration analysis of nanobeams including surface stress effects. *Applied Mathematics and Computation*, 219(10): 4977-4991, 2013.
11. R. Ansari, R. Gholami, K. Hosseini, and S. Sahmani. A sixth-order compact finite difference method for vibrational analysis of nanobeams embedded in an elastic medium based on nonlocal beam theory. *Mathematical and Computer Modeling*, 54(11-12), 2577-2586, 2011.
12. S. Yalcinbas. Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations. *Applied Mathematics and Computation*, 127(2-3):195-206, 2002.
13. S. Yalcinbas and M. Sezer. The approximate solution of high-order linear Volterra-Fredholm integro-differential equations in terms of Taylor polynomials. *Applied Mathematics and Computation*, 112(2-3):291-308, 2000.
14. S. Shahmorad. Numerical solution of the general form linear Fredholm-Volterra integro-differential equations by the Tau method with an error estimation. *Applied Mathematics and Computation*, 167(2):1418-1429.
15. F. Hosseini Shekarabi, K. Maleknejad and R. Ezzati, Application of two-dimensional Bernstein polynomials for solving mixed Volterra-Fredholm integral equations. *Afrika Matematika*, 26(7):1237-1251, 2015.
16. E. Babolian, S. Bazm and P. Lima. Numerical solution of nonlinear two-dimensional integral equations using rationalized Haar functions. *Communications in Nonlinear Science and Numerical Simulation*, 16(3):1164-1175, 2011.
17. M.A. Abdelkawy, A.Z. M. Amin, A.H. Bhrawy, J.A. Machado and A.M. Lopes. Jacobi collocation approximation for solving multi-dimensional Volterra integral equations. *International Journal of Nonlinear Sciences and Numerical Simulation*, 18(5):411-425, 2017.
18. F. Mirzaee and E. Hadadiyan. Applying the modified block-pulse functions to solve the three-dimensional Volterra-Fredholm integral equations. *Applied Mathematics and Computation*, 265:759-767, 2015.
19. K. Maleknejad, J. Rashidinia and T. Eftekhari. Numerical solution of three-dimensional Volterra-Fredholm integral equations of the first and second kinds based on Bernstein's approximation. *Applied Mathematics and Computation*, 339:272-285, 2018.
20. M. Esmaeilbeigi, F. Mirzaee and D. Moazami. A meshfree method for solving multidimensional linear Fredholm integral equations on the hypercube domains. *Applied Mathematics and Computation*, 298:236-246, 2017.
21. K. Sadri, A. Amini and C. Cheng. Low cost numerical solution for three-dimensional linear and nonlinear integral equations via three-dimensional Jacobi polynomials. *Journal of Computational and Applied Mathematics*, 319:493-513, 2017.
22. H. Adibi and P. Assari. Chebyshev wavelet method for numerical solution of Fredholm integral equations of the first kind, *Mathematical Problems in Engineering*, 1-17, 2010.
23. P. Assari, H. Adibi and M. Dehghan. A meshless discrete Galerkin (MDG) method for the numerical solution of integral equations with logarithmic kernels. *Journal of Computational and Applied Mathematics*, 267:160-181, 2018.
24. R. Ketabchi, R. Mokhtari and E. Babolian. A new approach for solving Volterra integral equations using the reproducing kernel method. *International Journal of Industrial Mathematics*, 9(1):21-26, 2017.
25. Z. Mahmoodi, J. Rashidinia and E. Babolian. B-Spline collocation method for linear and nonlinear Fredholm and Volterra integro-differential equations. *Applicable Analysis*, 92(9):1787-1802, 2013.
26. F. Kamali and H. Saeedi. Generalized fractionalorder Jacobi functions for solving a nonlinear systems of fractional partial differential equations numerically. *Mathematical Methods in the Applied Sciences*, 41(8):3155-3174, 2018.
27. K. Parand and M. Delkhosh, An efficient numerical solution of nonlinear Hunter-Saxton equation. *Communications in Theoretical Physics*, 67(5):483, 2017.
28. M. Delkhosh. The conversion a Bessels equation to a self-adjoint equation and applications. *World Applied Sciences* Journal, 15(12):1687-1691, 2011.
29. M.Shamsi and M.Razzaghi. Solution of Hallen's integral equation using multiwavelets. *Computer Physics Communications*, 168(3):187-197, 2005.
30. T. Ismaeelpour, A. Askari Hemmat and H. Saeedi. B-spline operational matrix of fractional integration. *Optik*, 130:291-305, 2017.

31. K. E. Atkinson. *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, New York, 1997.
32. D. Mirzaei and M.Dehghan. A meshless based method for solution of integral equations. *Applied Numerical Mathematics*, 60(3):245-262, 2010.
33. P. Assari, H. Adibi and M. Dehghan. A meshless method based on the moving least squares (MLS) approximation for the numerical solution of two-dimensional nonlinear integral equations of the second kind on non-rectangular domains. *Numerical Algorithms*, 67(2):423-455, 2014.
34. P. Assari, H. Adibi and M. Dehghan. A meshless method for solving nonlinear two-dimensional integral equations of the second kind on non-rectangular domains using radial basis functions with error analysis. *Journal of Computational and Applied Mathematics*, 239:72-92, 2013.
35. P. Assari, H. Adibi and M. Dehghan. A numerical method for solving linear integral equations of the second kind on the non-rectangular domains based on the meshless method. *Applied Mathematical Modeling*, 37(22):9269-9294, 2013.
36. K. Parand, S. Latifi, M. M. Moayeri and M. Delkhosh. Generalized Lagrange Jacobi Gauss-Lobatto (GLJGL) collocation method for solving linear and nonlinear Fokker-Planck equations. *Communications in Theoretical Physics*, 69(5):519, 2018.
37. K. Parand and M. Delkhosh. Systems of nonlinear Volterra integro-differential equations of arbitrary order. *Boletim da Sociedade Paranaense de Matemtica*, 36(4):33-54, 2018.
38. K. Parand, A. Bahramnezhad and H. Farahani. A numerical method based on rational Gegenbauer functions for solving boundary layer flow of a Powell-Eyring non-Newtonian fluid. *Computational and Applied Mathematics*, 37(5):6053-6075, 2018.
39. A.H. Bhrawy, M.A.Zaky and D.Baleanu. New numerical approximations for space-time fractional Burgers equations via a Legendre spectral-collocation method. *Romanian Reports in Physics*, 67(2):340-349, 2015.
40. C. Allouch, P. Sablonniere and D. Sbibih, A collocation method for the numerical solution of a two dimensional integral equation using a quadratic spline quasi-interpolant. *Numerical Algorithms*, 62(3):445-468, 2013.
41. R. L. Burden, J. D. Faires. *Numerical Analysis*, Youngstown State University, Youngstown, 2001.
42. A. Borhanifar and K. Sadri. Numerical solution for systems of two dimensional integral equations by using Jacobi operational collocation method. *Sohag J. Math*, 1(1):15-26, 2014
43. H. Asgharzadeh and I. Borazjani. A Newton-Krylov method with an approximate analytical Jacobian for implicit solution of Navier-Stokes equations on staggered overset-curvilinear grids with immersed boundaries. *Journal of Computational Physics*, 331:227-256, 2017.
44. D. A. Knoll and D. E. Keyes. Jacobian-free Newton-Krylov methods: a survey of approaches and applications. *Journal of Computational Physics*, 193(2):357-397, 2004.
45. J.Chen and C. Vuik. Globalization technique for projected Newton-Krylov methods. *International Journal for Numerical Methods in Engineering*, 110(7):661-674. 2017.
46. K. Parand, H. Yari, and M. Delkhosh. Solving two-dimensional integral equations of the second kind on non-rectangular domains with error estimate. *Engineering with Computers*, 36(2):725-739, 2020.
47. Y. Saad and M. H. Schultz..GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems, *SIAM Journal on Scientific and Statistical Computing*, 7:856-870, 1986.

