Research Article ISSN Online: 2783-4859 **Computational Mathematics and Computer Modeling with Applications**

Received 18 Oct 2022
Accepted 2 Dec 2022

DOI: 10.52547/CMCMA.1.2.27
AMS Subject Classification: 45E10; 46E22

Reproducing kernel method for Abel's second kind singular integral equations

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Singular integral equations (SIEs) are often encountered in certain contact and fracture problems in solid mechanics. In this paper, we apply the reproducing kernel method (RKM) to give the approximate solution of Abel's second-kind singular integral equations. For solving this problem, difficulties lie in its singular term. In order to remove the singular term of the equation, an equivalent transformation is made. Solution representations are obtained in reproducing kernel Hilbert space. Numerical experiments show that our reproducing kernel method is efficient. To show the high accuracy of the method the results are compared to other numerical methods and satisfactory agreements are achieved. Copyright © 2022 Shahid Beheshti University.

Keywords: Abel integral equation; Reproducing Hilbert kernel space; Approximate solution.

1. Introduction

In the present paper, we consider the following Abel integral equation of the second kind

$$y(x) = g(x) + Hy(x), \tag{1}$$

with

$$Hy(x) = \lambda \int_0^x \frac{N[y(t)]}{(x-t)^\alpha} dt, \qquad 0 \le x, t \le 1, \tag{2}$$

where $\lambda \in \mathbb{C}$ is a parameter, $g(x)$ is a non-homogeneous component, $y(x)$ is unknown function and has to be determined, $N[y(x)]$ is linear or nonlinear function of $y(x)$ in $W_2^1[0,1]$ and $y(x), g(x) \in W_2^1[0,1]$. For $\alpha \in (0,1)$, (1) is weakly singular Volterra integral equation that the special case $\alpha = \frac{1}{2}$ often occurs in physical problems. Abel integral equations have many applications in the various fields, such as solid mechanics, chemistry, scattering theory, astrophysics, fluid flow and heat condition [9, 14].

In recent years different techniques have been proposed for obtaining the approximate analytic and numeric solution of Abel integral equations. For instance Laplace decomposition method [8], Homotopy perturbation method [6], Adomian decomposition method [3], Chebyshev wavelets method [13], Bernstein operational method [2, 12], Taylor expansion [7], Block-pulse function method [11] and Variational iteration method [10]. The aim of this paper is implementation the theory of reproducing kernel [1, 4] to represent the approximate solution of Abel integral equation (1) in the reproducing kernel space.

The outline of this paper is as follows: In the next section we introduce an algorithm to represent the approximate solution of (1). In section 3 we present some numerical examples.

2. Method of solution

In this section, the solution of (1) is given in the space $W_2^1[0,1]$, is defined by

$$W_2^1[0,1] = \{y(x) | y \textit{ is absolutely continuous}, y, y' \in L^2[0,1]\}.$$

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In [4] it has been proved that $W_2^1[0,1]$ is a reproducing kernel space and its reproducing kernel is $R_t(x)$ where

$$R_t(x) = \begin{cases} 1+x, & x \le t, \\ 1+t, & x > t. \end{cases} \tag{3}$$

Let $S = \{x_1, x_2, \cdots\}$ is dense in the interval $[0,1]$. Put $\psi_i(x) = R_{x_i}(x)$. From the property of the reproducing kernel, it holds

$$\langle y(x), \psi_i(x)\rangle_{W_2^1} = y(x_i).$$

Theorem 1 *For* (1)*,* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete system of* $W_2^1[0,1]$ *if* S *be dense in* $[0,1]$.

Proof. Clearly $\psi_i(x) \in W_2^1[0,1]$. For each $y(x) \in W_2^1[0,1]$, let $\langle y(x), \psi_i(x)\rangle_{W_2^1} = 0$ $(i = 1, 2, \cdots)$ thus $y(x_i) = 0$. Since $y(x)$ is continuous and S is dense in $[0,1]$ then $y(x) = 0$ for each $x \in [0,1]$. So proof is complete. □

We apply Gram-Schmidt ortho-normalization to derive the orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x),$$

where β_{ik} are coefficients of Gram-Schmidt ortho-normalization.

Theorem 2 *Let* S *be dense in* $[0,1]$*. If* $y(x)$ *be a unique solution of* (1) *then* $y(x)$ *satisfies the form*

$$y(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(g(x_k) + Hy(x_k))\overline{\psi}_i(x). \tag{4}$$

Proof. Assume $y(x)$ be the solution of (1) and $y(x) \in W_2^1[0,1]$. From Theorem 1, $y(x)$ can be expanded to Fourier series as follow,

$$\begin{aligned} y(x) &= \sum_{i=1}^{\infty} \langle y(x), \overline{\psi}_i(x)\rangle_{W_2^1}\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle y(x), \psi_k(x)\rangle_{W_2^1}\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\langle g(x) + Hy(x), \psi_k(x)\rangle_{W_2^1}\overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(g(x_k) + Hy(x_k))\overline{\psi}_i(x). \end{aligned}$$

□

Note that the approximate solution $y_n(x)$ of (1) can be obtained by

$$y_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}(g(x_k) + Hy(x_k))\overline{\psi}_i(x). \tag{5}$$

Theorem 3 *The approximate solution* $y_n(x)$ *uniformly converges to the exact solution* $y(x)$.

Proof. Let $R_x(t)$ is the reproducing kernel of the space $W_2^1[0,1]$. Because $R_x(t)$ is continuous in $[0,1]$, so there exist a positive constant M such that $\| R_x(t) \|_{W_2^1} \le M$. Then

$$\begin{aligned} | y_n(x) - y(x) | &= | \langle y_n(t) - y(t), R_x(t)\rangle_{W_2^1} | \\ &\le \| y_n(t) - y(t) \|_{W_2^1} \| R_x(t) \|_{W_2^1} \\ &\le M \| y_n(t) - y(t) \|_{W_2^1} . \end{aligned}$$

We know $\| y_n(t) - y(t) \|_{W_2^1} \longrightarrow 0$ *as* $n \longrightarrow \infty$ *so* $| y_n(x) - y(x) | \longrightarrow 0$ *as* $n \longrightarrow \infty$. The proof is complete. □

The approximate solution y_n in (5) is suitable for the case $N[y(x)]$ in (2) be linear. If $N[y(x)]$ be a nonlinear term we have to use the following iteration formula to get approximate solution.

$$\begin{cases} y_1(x) = g(x), \\ y_{n+1}(x) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x), \end{cases} \tag{6}$$

where the coefficients A_i are unknown and given as

$$\begin{cases} A_1 = \beta_{11}(g(x_1) + Hy_1(x_1)), \\ A_2 = \sum_{k=1}^{2} \beta_{2k}(g(x_k) + Hy_2(x_k)), \\ \cdots \\ A_n = \sum_{k=1}^{n} \beta_{nk}(g(x_k) + Hy_n(x_k)). \end{cases} \tag{7}$$

Lemma 1 *If $\|y_n(x) - \overline{y}(x)\|_{W_2^1} \longrightarrow 0$ and $x_n \longrightarrow t$ as $n \longrightarrow \infty$, then $y_n(x_n) \longrightarrow \overline{y}(t)$ as $n \longrightarrow \infty$.*

Proof. Since

$$\begin{aligned} \mid y_n(x_n) - \overline{y}(t) \mid &= \mid y_n(x_n) - y_n(t) + y_n(t) - \overline{y}(t) \mid \\ &\leq \mid y_n(x_n) - y_n(t) \mid + \mid y_n(t) - \overline{y}(t) \mid, \end{aligned}$$

by reproducing kernel property of $R_x(t)$, we have $y_n(x_n) = \langle y_n(x), R_{x_n}(x)\rangle_{W_2^1}$ and $y_n(t) = \langle y_n(x), R_t(x)\rangle_{W_2^1}$, hence

$$\begin{aligned} \mid y_n(x_n) - y_n(t) \mid &= \mid \langle y_n(x), R_{x_n}(x) - R_t(x)\rangle_{W_2^1} \mid \\ &\leq \|y_n(x)\|_{W_2^1} \|R_{x_n}(x) - R_t(x)\|_{W_2^1}. \end{aligned}$$

From symmetry of $R_t(x)$, it follows that $\|R_{x_n}(x) - R_t(x)\|_{W_2^1} \longrightarrow 0$ as $n \longrightarrow \infty$.
So $\mid y_n(x_n) - y_n(t) \mid \longrightarrow 0$ as soon as $x_n \longrightarrow t$. For any $t \in [0, 1]$, it holds that $\mid y_n(t) - \overline{y}(t) \mid \longrightarrow 0$.
Therefore $\mid y_n(x_n) - \overline{y}(t) \mid \longrightarrow 0$ as $n \longrightarrow \infty$. The proof is complete. □

Theorem 4 *If S be dense in $[0, 1]$ and $\|y_n(x)\|_{W_2^1}$ be bounded, then the n-term approximate solution $y_n(x)$ of (6) is convergence to the exact solution $y(x)$ of (1) and*

$$y(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x),$$

where A_i is given by (7).

Proof. (1) We will prove the convergence of (6). By (7), we have

$$y_{n+1}(x) = y_n(x) + A_n \overline{\psi}_n(x). \tag{8}$$

From the orthogonality of $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ we can write
$\|y_{n+1}(x)\|^2_{W_2^1} = \|y_n(x)\|^2_{W_2^1} + A_n^2$.
The sequence $\|y_n\|_{W_2^1}$ is monotone increasing. Because of the condition that $\|y_n\|_{W_2^1}$ is bounded, $\|y_n\|_{W_2^1}$ is convergence as soon as $n \longrightarrow \infty$. So there exists a constant c such that

$$\sum_{i=1}^{\infty} A_i^2 = c.$$

It implies that

$$A_i = \sum_{k=1}^{i} \beta_{ik}(g(x_k) + Hy_i(x_k)) \in \ell^2, \; i = 1, 2, \cdots$$

Let $m > n$, from the orthogonality of $y_{n+1}(x) - y_n(x), \; n = 2, 3, \cdots$, we have

$$\begin{aligned} \|y_m(x) - y_n(x)\|^2_{W_2^1} &= \|y_m(x) - y_{m-1}(x) + y_{m-1}(x) - y_{m-2}(x) + \cdots + y_{n+1}(x) - y_n(x)\|^2_{W_2^1} \\ &\leq \|y_m(x) - y_{m-1}(x)\|^2_{W_2^1} + \|y_{m-1}(x) - y_{m-2}(x)\|^2_{W_2^1} + \cdots + \|y_{n+1}(x) - y_n(x)\|^2_{W_2^1} \\ &= \sum_{i=n+1}^{m} A_i^2. \end{aligned}$$

As $n \longrightarrow \infty$, we have $\sum_{i=n+1}^{m} A_i^2 \longrightarrow 0$, so $\|y_m(x) - y_n(x)\|^2_{W_2^1} \longrightarrow 0$.
From the completeness of $W_2^1[0,1]$, there exists a $\overline{y}(x) \in W_2^1[0,1]$ such that $\|y_n(x) - \overline{y}(x)\| \longrightarrow 0$.
Hence

$$\overline{y}(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x). \tag{9}$$

(2). We will prove that $y_{n+1}(x_k) = \overline{y}(x_k)$ for all $k \le n$. Define the orthogonal projection P_n from $W_2^1[0,1]$ to $Span\{\overline{\psi}_1, \overline{\psi}_2, \cdots, \overline{\psi}_n\}$ by

$$P_n \overline{y}(x) = \sum_{i=1}^{n} A_i \overline{\psi}_i(x).$$

We have

$$\begin{aligned} y_{n+1}(x_k) =& \langle y_{n+1}(x), \psi_k(x)\rangle_{W_2^1} \\ =& \langle P_n \overline{y}(x), \psi_k(x)\rangle_{W_2^1} \\ =& \langle \overline{y}(x), P_n \psi_k(x)\rangle_{W_2^1} \\ =& \langle \overline{y}(x), \psi_k(x)\rangle_{W_2^1} \\ =& \overline{y}(x_k). \end{aligned}$$

Thus,

$$H y_{n+1}(x_k) = H\overline{y}(x_k).$$

(3). In this step, we will prove that $\overline{y}(x)$ is the solution of (1).
From (9), it follows

$$\overline{y}(x_j) = \sum_{i=1}^{\infty} A_i \langle \overline{\psi}_i(x), \psi_j(x)\rangle_{W_2^1}. \tag{10}$$

Multiplying both side of (10) by β_{nj} and summing for j from 1 to n. Since $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ has orthogonality property, we have

$$\begin{aligned} \sum_{j=1}^{n} \beta_{nj} \overline{y}(x_j) =& \sum_{i=1}^{\infty} A_i \langle \overline{\psi}_i(x), \sum_{j=1}^{n} \beta_{nj} \psi_j(x)\rangle_{W_2^1} \\ =& \sum_{i=1}^{\infty} A_i \langle \overline{\psi}_i(x), \overline{\psi}_n(x)\rangle_{W_2^1} = A_n. \end{aligned}$$

Now, if $n = 1$, then

$$\beta_{11} \overline{y}(x_1) = A_1 = \beta_{11}(g(x_1) + H y_1(x_1)),$$

on the other hand, if $n = 2$ then

$$\beta_{21} \overline{y}(x_1) + \beta_{22} \overline{y}(x_2) = A_2 = \beta_{21} \overline{y}(x_1) + \beta_{22}(g(x_2) + H y_2(x_2)),$$

Thus

$$\overline{y}(x_2) = g(x_2) + H y_2(x_2).$$

It easy to see that

$$\overline{y}(x_n) = g(x_n) + H y_n(x_n).$$

For any $x \in [0,1]$, since $S = \{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$ there exists a subsequence $\{x_{n_k}\}_{i=1}^{\infty}$ such that $x_{n_k} \longrightarrow x$.
From Lemma 2.4 and the above form, we have

$$\overline{y}(x) = g(x) + H y(x).$$

That is, $y(x)$ is the solution of (1). The proof is complete. □

Theorem 5 *Assume that $y(x)$ is the solution of (1) and $r_n(x)$ is the error of the approximate solution $y_{n+1}(x)$ that is given by (6), then $r_n(x)$ is monotone decreasing in the sense of $\|.\|_{W_2^1}$, i.e $r_n \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof. We know that

$$\|r_n(x)\|^2_{W_2^1} = \|y(x) - y_{n+1}(x)\|^2_{W_2^1} = \|\sum_{i=n+1}^{\infty} A_i\overline{\psi_i}(x)\|^2_{W_2^1} = \sum_{i=n+1}^{\infty} A_i^2,$$

$$\|r_{n-1}(x)\|^2_{W_2^1} = \sum_{i=n}^{\infty} A_i^2.$$

Thus $\|r_n(x)\|^2_{W_2^1} \leq \|r_{n-1}(x)\|^2_{W_2^1}$. Consequently, $r_n(x)$ is monotone decreasing in the sense of $\|.\|_{W_2^1}$. □

3. Numerical examples

In this section three numerical examples are presented to give a clear overview of the procedure. For all of these examples the exact solutions are available.

Example 1. As the first example, consider (1)-(2) with [11, 13]

$$N[y(x)] = y(x), \; g(x) = \frac{1}{\sqrt{1+x}} - \frac{1}{4}\arcsin(\frac{1-x}{1+x}) + \frac{\pi}{8}, \; \lambda = \frac{-1}{4}, \alpha = \frac{1}{2},$$

where the exact solution is $y(x) = \frac{1}{\sqrt{1+x}}$. Table 1 shows the absolute values of error for $y(x)$ with $n = 20$, using the method proposed in Section 2 and compares the result with result obtained by various numerical schemes at different gird point. As we can see the proposed method is much more accurate than other methods. Figure 1 shows the comparison with exact solution for $n = 20$.

Table 1. Absolute values of error for y from Example 1 with $n = 20$

x	Presented method	method[13] (k=0,M=16)	method[11] (m=16)
0	0.0	–	–
0.1	3.33067×10^{-16}	8.09355×10^{-14}	3×10^{-3}
0.2	2.22045×10^{-16}	1.09024×10^{-13}	1×10^{-3}
0.3	3.33067×10^{-16}	1.38001×10^{-13}	1×10^{-3}
0.4	3.33067×10^{-16}	1.33005×10^{-13}	2×10^{-3}
0.5	5.55112×10^{-16}	1.28009×10^{-13}	4×10^{-3}
0.6	4.44089×10^{-16}	1.22014×10^{-13}	2×10^{-3}
0.7	3.33067×10^{-16}	1.09912×10^{-13}	6×10^{-4}
0.8	2.22045×10^{-16}	6.29496×10^{-14}	6×10^{-4}
0.9	7.77156×10^{-16}	8.69305×10^{-14}	1×10^{-3}
1	4.44089×10^{-16}	–	–

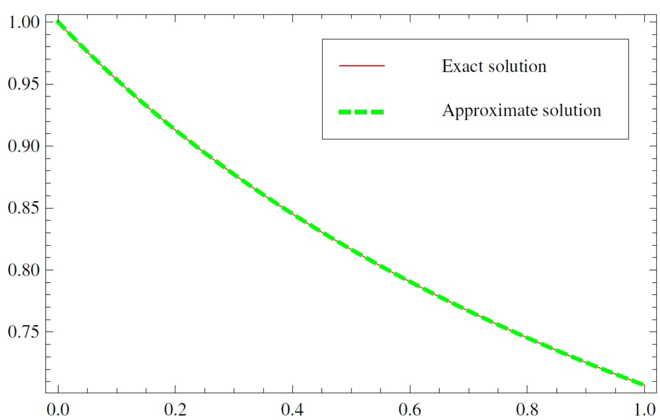

Figure 1. Comparison among exact solution and approximate solution for Example 1 with $n = 20$.

Example 2. In this example, consider (1)-(2) with [11, 13, 15]

$$N[y(x)] = y(x),\ \ g(x) = 2\sqrt{x},\ \ \lambda = -1, \alpha = \frac{1}{2},$$

the exact solution for this example is $y(x) = 1 - e^{\pi x} erfc(\sqrt{\pi x})$, where $erfc(\sqrt{\pi x})$ is complementary error function defined by $erfc(x) = \frac{2}{\sqrt{x}} \int_x^\infty e^{-t^2} dt$.

The comparison among absolute values of error is shown in Table 2. Figure 2 shows the comparison with exact solution for $n = 50$.

Table 2. Absolute values of error for y from Example 2 with $n = 20$

x	Presented method	method[15] [4/4]	method[13] (k=0,M=16)	method[11] (m=16)
0	2.45659×10^{-16}	–	–	–
0.1	1.66533×10^{-16}	4.33846×10^{-9}	1.62983×10^{-4}	1.15872×10^{-2}
0.2	1.11022×10^{-16}	4.87786×10^{-8}	2.82352×10^{-4}	1.13995×10^{-2}
0.3	0.0	1.82276×10^{-7}	1.89633×10^{-4}	9.55367×10^{-3}
0.4	1.11022×10^{-16}	4.42272×10^{-7}	1.43922×10^{-4}	1.68375×10^{-3}
0.5	2.22045×10^{-16}	8.53771×10^{-7}	1.32002×10^{-4}	7.61903×10^{-3}
0.6	0.0	1.43214×10^{-6}	1.21446×10^{-4}	1.53846×10^{-3}
0.7	0.0	2.18575×10^{-6}	9.86938×10^{-5}	3.09894×10^{-4}
0.8	0.0	3.11805×10^{-6}	2.45968×10^{-5}	2.98197×10^{-4}
0.9	1.11022×10^{-16}	4.22898×10^{-6}	1.45968×10^{-4}	7.08482×10^{-4}
1	0	–	–	–

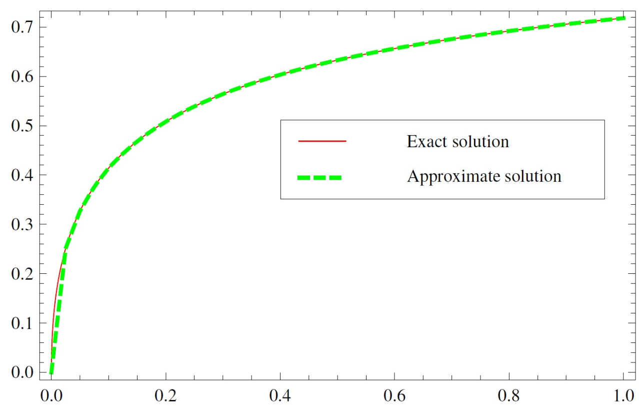

Figure 2. Comparison among exact solution and approximate solution for Example 2 with $n = 50$.

Example 3. As the last example, we consider the nonlinear case for (1)-(2), with [5]

$$N[y(x)] = y^2(x),\ \ g(x) = 1 - x^2 + \frac{2\sqrt{x}}{315}(315 - 336x^2 + 128x^4),\ \ \lambda = -1,\ \alpha = \frac{1}{2},$$

which has the exact solution $y(x) = 1 - x^2$. Figure 3 shows the comparison with exact solution for $n = 20$. The comparison among absolute values of error is shown in Table 3.

4. Conclusions

In the current work, we have revisited Abel integral equations which have many applications in various fields, such as solid mechanics, chemistry, scattering theory, astrophysics, fluid flow, and heat condition. Then, the reproducing kernel method (RKM) has been applied to represent an approximate solution in reproducing kernel Hilbert space. Some numerical illustrative examples have been given to show the efficiency and high accuracy of the method in comparison to the other recently proposed methods such as radial basis and Block-Pulse functions collocation techniques.

Table 3. Absolute values of error for y from Example 3

x	n=10	n=20
0	2.22045×10^{-16}	2.22045×10^{-16}
0.1	3.33067×10^{-16}	6.66134×10^{-16}
0.2	1.11022×10^{-16}	8.88178×10^{-15}
0.3	3.33067×10^{-16}	1.11022×10^{-16}
0.4	1.11022×10^{-16}	4.44089×10^{-16}
0.5	1.11022×10^{-15}	1.11022×10^{-16}
0.6	4.44089×10^{-16}	1.11022×10^{-15}
0.7	3.33067×10^{-16}	4.44089×10^{-16}
0.8	9.99201×10^{-16}	1.38778×10^{-15}
0.9	0.0	1.11022×10^{-15}
1	0.0	6.66134×10^{-16}

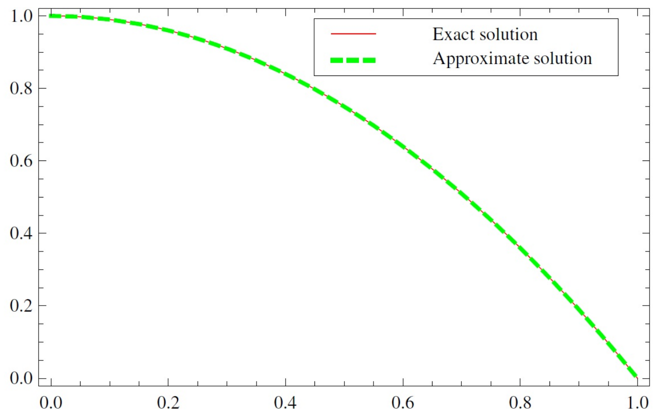

Figure 3. Comparison among exact solution and approximate solution for Example 3 with $n = 20$.

Acknowledgments

We thank the anonymous reviewers for helpful comments, which lead to definite improvement in the manuscript.

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