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A class of efficient derivative free iterative method with and without memory for solving nonlinear equations

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In the present paper, at first, we propose a new two-step iterative method for solving nonlinear equations. This scheme is based on the Steffensen's method, in which the order of convergence is four. This iterative method requires only three functions evaluation in each iteration, therefore it is optimal in the sense of the Kung and Traub conjecture. Then we extend it to the method with memory, which the order of convergence is six. Finally, numerical examples indicate that the obtained methods in terms of accuracy and computational cost are superior to the famous forth-order methods. Copyright © 2022 Shahid Beheshti University.

Keywords: Nonlinear equations; Two-step methods; Efficiency index; Order of convergence; Simple root; Iterative method with memory.

1. Introduction

In recent years, solving nonlinear equations has attracted the attention of many researchers, since many of the complexities in science, engineering and optimization include nonlinear functions in the equation of the form

$$f(x) = 0, \tag{1}$$

where $f : D \to \mathbb{R}$, and D is an open interval.

In this study, we examine to find the simple root. It means, If $\alpha \in D$, is a simple root of f, then $f(\alpha) = 0$, and $f'(\alpha) \neq 0$. Iterative methods are often used to obtain an approximation of solution of problem. As we know, Newton's method is the most famous iterative method for finding α, by the scheme:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \tag{2}$$

which converges quadratically in some neighborhood of α [13, 18].

Since the derivative of the function may not be defined in every interval, Newton's method can not be always used. Therefore, Steffensen modified Newton's method, replaced the derivative $f'(x_k)$, with the difference approximation:

$$f'(x_k) = \frac{f(x_k + \beta f(x_k)) - f(x_k)}{\beta f(x_k)},$$

where $\beta \in \mathbb{R} - \{0\}$. Therefore, Steffensen's method as follows [3]:

$$x_{k+1} = x_k - \frac{\beta f(x_k)^2}{f(x_k + \beta f(x_k)) - f(x_k)}. \tag{3}$$

Because the use of the iterative methods which depend on derivatives creates some limitations on engineering applications, derivative free methods are particular importance.

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Researchers in the field of numerical analysis have proposed various methods [2, 19, 5, 14] to improve the order of convergence of Newton's method and Steffensen's method by adding the cost of evaluating functions, derivatives and changes in the points of iterations. All of these modifications are aimed at increasing the order of convergence, and so increasing their efficiency indices.

In the following, we give some definitions which describe some properties about iterative methods.

Definition 1 *[18] Assume that α is a simple root of a real function f, and $\{x_k\}_{k\geq 0}$, is a sequence of real numbers, converging to α. Then, the order of convergence of the sequence is p, if there is a constant real number $C \neq 0$, such that*

$$\lim_{x\to\infty} \frac{|x_{k+1}-\alpha|}{|x_k-\alpha|^p} = C,$$

C is called the asymptotic error constant.

Definition 2 *[18] If we show the error in the k-th iteration with $e_k = x_k - \alpha$, then the relation*

$$e_{k+1} = Ce_k^p + \mathcal{O}(e_k^{p+1}),$$

is called the error equation.

By examining the error equation for any iterative method, the value of p and C would be, respectively, the convergence order and the asymptotic error constant.

Definition 3 *[18] The efficiency index (EI) of a method is defined by:*

$$EI = p^{\frac{1}{c}},$$

where p and c are the order of convergence and the total number of functions evaluations per every iteration, respectively.

In many of the high-convergence multi-step methods presented in recent years, inverse, Hermite, and rational interpolations [8, 11, 6] have been used. In which, the Kung and Traub conjecture has been applied for the development of these methods. This conjecture is as follows:

Conjecture 1. *[9] An optimal iterative method without memory based on k function evaluations would achieve an optimal convergence order of 2^{k-1}. Therefore, $EI = 2^{\frac{(k-1)}{k}}$.*

Ostrowski's method as one of the famous fourth-order methods without memory is as follows [8]:

$$\begin{cases} y_k = x_k - \dfrac{f(x_k)}{f'(x_k)}, \\[2ex] x_{k+1} = y_k - \dfrac{f(x_k)}{f(x_k)-2f(y_k)}\dfrac{f(y_k)}{f'(x_k)}. \end{cases} \tag{4}$$

This method requires two functions of f and one derivative f' evaluations in each step, therefore $EI = \sqrt[3]{4}$.

Also, another fourth-order methods without memory is Jarrat's method [11], which is defined as:

$$\begin{cases} y_k = x_k - \dfrac{2f(x_k)}{3f'(x_k)}, \\[2ex] x_{k+1} = x_k - \dfrac{3f'(y_k)+f'(x_k)}{6f'(y_k)-2f'(x_k)}\dfrac{f(x_k)}{f'(x_k)}. \end{cases} \tag{5}$$

The method (5) requires one function f and two derivative f' evaluations in each iteration.

2. A new class fourth-order iterative method

In this section, we present a new method by using only three functions evaluations. The Steffensen's method is the first step, and the second step is constructed in a way that the order of convergence of the two step method becomes four. The new proposed method is as follows:

$$\begin{cases} y_k = x_k - \dfrac{\beta f(x_k)^2}{f(w_k)-f(x_k)}, \qquad w_k = x_k + \beta f(x_k), \\[2ex] x_{k+1} = y_k - \Big(\dfrac{1}{f(x_k)-f(y_k)(\frac{f(x_k)}{f(w_k)-f(y_k)}+1)} - \dfrac{f(y_k)^2}{f(w_k)^2 f(x_k)}\Big)(x_k-y_k)f(y_k). \end{cases} \tag{6}$$

In the next theorem, we show that the method (6) is fourth-order convergent.

Theorem 1 *Let $\alpha \in D$ be a simple root of (1) and the initial point x_0 is sufficiently close to α, then the method (6) defines a family of fourth-order convergent methods with the following error equation:*

$$e_{k+1} = (1+\beta f'(\alpha))^2 c_2(c_2^2 - c_3)e_k^4 + \mathcal{O}(e_k^5),$$

where $e_k = x_k - \alpha$, and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k \geq 2$.

Proof. According to Taylor expansion for the function f, we have

$$f(x_k) = f'(\alpha)\Big(e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4 + \mathcal{O}(e_k^5)\Big), \tag{7}$$

and

$$f(x_k + \beta f(x_k)) = f'(\alpha)\Big((1+\beta f'(\alpha))e_k + (1+\beta f'(\alpha)(3+\beta f'(\alpha))c_2e_k^2 + \mathcal{O}(e_k^3)\Big). \tag{8}$$

By dividing (7) and (8), we can obtain

$$\frac{\beta f(x_k)^2}{f(w_k) - f(x_k)} = e_k - (1+\beta f'(\alpha))c_2e_k^2 + \mathcal{O}(e_k^3). \tag{9}$$

Then, we have

$$y_k = (1+\beta f'(\alpha))c_2e_k^2 + \mathcal{O}(e_k^3), \tag{10}$$

so the result is

$$f(y_k) = f'(\alpha)\Big((1+\beta f'(\alpha))c_2e_k^2 + \mathcal{O}(e_k^3)\Big). \tag{11}$$

Similarly, for the second step of (6), we have

$$\Big(\frac{1}{f(x_k) - f(y_k)(\frac{f(x_k)}{f(w_k)-f(y_k)}+1)} - \frac{f(y_k)^2}{f(w_k)^2 f(x_k)}\Big)(x_k - y_k)f(y_k) = (1+\beta f'(\alpha))c_2e_k^2 + \mathcal{O}(e_k^3).$$

Finally,

$$x_{k+1} - \alpha = (1+\beta f'(\alpha))^2 c_2(c_2^2 - c_3)e_k^4 + \mathcal{O}(e_k^5),$$

that means the method (6) has order of convergence four.

□

Note that the optimal two-step four order convergent method requires only three functions evaluations in each iteration, therefore, $EI = \sqrt[3]{4} \approx 1.587$, which is in agreement with the conjecture of Knug and Traub for $k = 3$.

3. Development of the new method with memory and convergence analysis

In recent years, researchers have proposed various papers to improve the order of the without memory methods to the the with memory method [10, 12, 17, 7, 15, 1, 16, 4]. This section represents the construction of a method with memory from (6) by using a self-accelerating parameter. According to Theorem (1), the order of method (6) is four, if $\beta \neq -\frac{1}{f'(\alpha)}$. In fact, if $\beta = -\frac{1}{f'(\alpha)}$, then, at least the order of convergence is four. Since the exact value of (simple) zero is not available, $f'(\alpha)$ can not be computed. On the other hand, we can approximate $f'(\alpha)$ with available data.

The main idea of constructing methods with memory is to compute the parameter $\beta = \beta_k$, as the iteration proceeds by the formulas $\beta_k = -\frac{1}{\hat{f}'(\alpha)}$, for $k = 1, 2, \ldots$, where $\hat{f}'(\alpha)$ is approximated by $f'(\alpha)$. To this end, the following formula is applied:

$$\beta_k = -\frac{1}{\hat{f}'(\alpha)} = -\frac{1}{N_3'(x_k)},$$

where $N_3(t) = N_3(t; x_k, x_{k-1}, y_{k-1}, w_{k-1})$, is the Newton's interpolating polynomial of third degree, which is set through the best approximations available for the nodel point $(x_k, y_{k-1}, w_{k-1}, x_{k-1})$. By replacing the fixed parameter β, in the iterative formula (6) with β_k, we present the following new method with memory (x_0, β_0, are given):

$$\begin{cases} \beta_k = -\frac{1}{\hat{f}'(\alpha)} = -\frac{1}{N_3'(x_k)}, \qquad w_k = x_k + \beta_k f(x_k), \\[2ex] y_k = x_k - \frac{\beta_k f(x_k)^2}{f(w_k) - f(x_k)}, \\[2ex] x_{k+1} = y_k - \Big(\frac{1}{f(x_k) - f(y_k)(\frac{f(x_k)}{f(w_k)-f(y_k)}+1)} - \frac{f(y_k)^2}{f(w_k)^2 f(x_k)}\Big)(x_k - y_k)f(y_k). \end{cases} \tag{12}$$

Note that this idea is repeated in every step based on changes in the non-zero free parameter β_n. The calculation of this parameter has been done by using the information obtained from the current and previous iteration(s), therefore, following Traub's classification, the extentioned method can be considered as a method with memory [18].

The error relations with the β_k self-accelerating parameter for methods are symbolic in the following cases:

$$e_{k,w} = w_k - \alpha \sim c_{k,1} e_k, \quad e_{k,y} = y_k - \alpha \sim c_{k,2} e_k^2, \quad e_{k+1} = x_{k+1} - \alpha \sim c_{k,4} e_k^4.$$

Now, we prove that the new method (12) with memory have the order of convergence 6. For this purpose, we have the following lemma.

Lemma 1 *Let $\beta_k = \dfrac{-1}{N_3(x_k)}$, then the following asymptotic relation is established:*

$$1 + \beta_k f'(\alpha) \sim c_4 e_{k-1} e_{k-1,y} e_{k-1,w}, \tag{13}$$

where $e_k = x_k - \alpha$, $e_{k,y} = y_k - \alpha$ and $e_{k,w} = w_k - \alpha$.

Proof. Assume that $\{x_k\}$ is a sequence of approximations generated by (12). At first, we must find the asymptotic error constants for the self-accelerator parameter. Then the substitution $\beta_k = -\dfrac{1}{f'(\alpha)}$. We calculate Newton's interpolating polynomial of $(x_{k-1}, f(x_{k-1}))$, $(w_{k-1}, f(w_{k-1}))$, $(y_{k-1}, f(y_{k-1}))$, and $(x_k, f(x_k))$, then $-\dfrac{1}{\widehat{f}'(x_k)}$. But since calculation is long at this stage, we write a piece of Mathematica codes as follows:

```
Clear All[" Global '*"]
A[t_]:= InterpolatingPolynomial[{{e,fe},{ew, few},{ey, fey},{e1, fe1},}, t]
Approximation = -1/A'[e] // Simplify;
fe=f1a*(e+c2*e^2+c3*e^3+c4*e^4);
fe1=f1a*(e1+c2*e1^2+c3*e1^3+c4*e1^4);
few=f1a*(ew+c2*ew^2+c3*ew^3+c4*ew^4);
fey=f1a*(ey+c2*ey^2+c3*ey^3+c4*ey^4);
β =Series[Approximation,{e,0,2},{ew,0,2},{ey,0,2},{e1,0,0}] //Simplify;
Collect[Series[(1+β*f1a ,{e,0,1},{ew,0,1},{ey,0,1},{e1,0,0}],
{e,ew,ey,e1},Simplify]
```

which results in

$$1 + \beta_k f'(\alpha) \sim c_4 e_{k-1} e_{k-1,y} e_{k-1,w}.$$

□

Theorem 2 *Suppose that the function $f(x)$ is sufficiently differentiable in a neighborhood of its simple zero α. If the initial guess is x_0, sufficiently close to α. Then, the order of convergence of the two-step method (12) with memory is at least 6.*

Proof. According to Lemma (1), we have

$$e_{k+1} \sim c_2(c_2^2 - c_3) e_{k-1}^2 e_{k-1,y}^2 e_{k-1,w}^2 e_k^4,$$

therefore, we can get

$$e_{k+1} \sim c_2(c_2^2 - c_3) e_{k-1}^8 e_k^4,$$

and then we simply obtain

$$e_{k+1} \sim c_2(c_2^2 - c_3) e_k^6.$$

The proof is complete.

□

4. Efficiency index

In the present study, we have extended a proposed two-step optimal sixth-order scheme, which is a method with memory. The EI of the method (12) is $6^{\frac{1}{3}} \approx 1.8171$.

If we show the methods (6), (12) with $IM1$ and $IM2$ respectively, as a method with and without memory, then the EI could be given as follows:

$$EI_{IM1} = \frac{s(4^{\frac{1}{3}})}{s}, \quad EI_{IM2} = \frac{4^{\frac{1}{3}} + (s-1)6^{\frac{1}{3}}}{s},$$

where s is the number of iterations. Figure 1 indicates the difference, and clearly shows the superiority of method $IM2$.

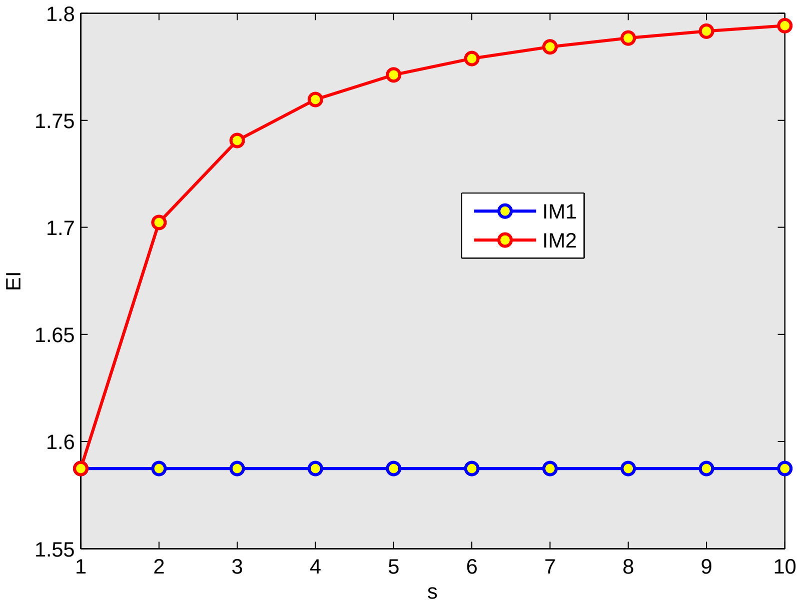

Figure 1. Comparison of EI for different methods

5. Numerical results

In this section, we present the results of numerical simulations to compare the order of convergence of the methods (6) and (12). Numerical computations reported are run in MATLAB software. The computational order of convergence is defined by [6]:

$$r_c = \frac{ln(|x_{n+2} - x_{k+1}|/|x_{k+1} - x_k|)}{ln(|x_{k+1} - x_k|/|x_k - x_{k-1}|)}. \tag{14}$$

For numerical tests, we use the functions that are in most articles as follows:

$$\begin{aligned}
&f_1(x) = x^3 - 10, && \alpha \approx 2.154434690031884,\\
&f_2(x) = (\sin x)^2 - x^2 + 1, && \alpha \approx 1.404491648215341,\\
&f_3(x) = \sqrt{x^2 + 2x + 5} - 2\sin x - x^2 + 3, && \alpha \approx 2.331969765588396,\\
&f_4(x) = x^3 - 3x^2 + x - 2, && \alpha \approx 2.893289196304498,\\
&f_5(x) = 2\sin x + 1 - x, && \alpha \approx 2.380061273139339,\\
&f_6(x) = e^{-x} + \cos x, && \alpha \approx 1.746139530408013,\\
&f_7(x) = \cos^2 x - \frac{x}{5}, && \alpha \approx 2.320204274495726.
\end{aligned}$$

Tables 1 and 2, show the results of the behavior for the functions. Also the r_c of each method indicates the consistency of the order of convergence of the theory and the experiment. To compare method with memory, we consider $\beta_0 = 0.01$. In all tables, we used the notation 5.35E-3, to show that, $|f_1(x_1)| \leq 5.35 \times 10^{-3}$, when $x_0 = 2.5$.

By applying one different initial guesse x_0, we implement only four full iterations of above various presented methods and compute $|f_4(x)|$. According to Table 3, it can be said that the results given by the proposed methods have high accuracy while the results given by methods (4) and (5) are desirable.

Note that in each of the methods, the use of f takes a certain CPU time, and since the calculation of f' is more than the calculation of f, it takes more CPU time. Also, it takes more time for methods with memory to store previous iterations, but we achieve better results with this modification.

Table 1. Numerical results of the iterative method (6) without memory for different test functions

function	x_0	$\|f(x_1)\|$	$\|f(x_2)\|$	$\|f(x_3)\|$	$\|f(x_4)\|$	r_c	CPU
f_1	2.5	5.35E-3	2.63E-14	1.58E-59	1.74E-240	3.9999	0.3701
f_2	3	2.73E-1	5.74E-5	2.79E-19	1.55E-76	3.9284	0.3193
f_3	1.4	6.41E-4	9.87E-17	5.54E-68	5.50E-273	4.0000	0.3864
f_4	2.6	1.36E-1	1.04E-7	4.18E-32	1.07E-129	3.9935	0.3240
f_5	2.2	2.10E-4	6.33E-18	5.23E-72	2.44E-288	3.9999	0.3071
f_6	1	2.69E-4	6.85E-17	2.89E-67	9.17E-269	4.0000	0.2971
f_7	2.1	1.15E-3	2.59E-13	6.87E-52	3.39E-206	3.9987	0.3169

Table 2. Numerical results of the iterative method (12) with memory for different test functions

function	x_0	β_0	$\|f(x_1)\|$	$\|f(x_2)\|$	$\|f(x_3)\|$	$\|f(x_4)\|$	r_c	CPU
f_1	2.5	2	5.35E-3	2.26E-21	1.28E-131	4.15E-798	5.9999	0.4518
f_2	3	2.5	2.73E-1	1.12E-6	1.36E-38	3.37E-230	5.9635	0.4942
f_3	1.4	1	6.41E-4	1.79E-25	4.59E-155	1.30E-932	6.0124	0.5779
f_4	2.6	2	1.36E-1	2.93E-11	3.25E-69	5.96E-417	5.9991	0.5119
f_5	2.2	2	2.10E-4	1.26E-27	6.00E-167	7.18E-1003	5.9992	0.4552
f_6	1	0.5	2.69E-4	4.93E-27	1.64E-162	2.29E-975	5.9584	0.4490
f_7	2.1	1.5	1.15E-3	1.87E-20	7.10E-122	2.38E-730	6.0411	0.5611

Table 3. Numerical results of several iterative methods

			Method(4)		Method(5)		Method(6)		Method(12)	
Function	x_0	x_{-1}	$\|f(x)\|$	CPU	$\|f(x)\|$	CPU	$\|f(x)\|$	CPU	$\|f(x)\|$	CPU
f_1	3	3.5	2.20E-142	0.4137	2.20E-142	0.4519	2.38E-159	0.4129	1.12E-518	0.5326
f_2	2	2.5	8.94E-126	0.3619	1.74E-127	0.3768	6.57E-141	0.3511	4.28E-455	0.4485
f_3	1.5	1	2.08E-296	0.4660	6.39E-222	0.4388	5.16E-301	0.3797	4.29E-1027	0.5233
f_4	3	3.5	2.80E-313	0.3693	2.80E-313	0.3787	3.86E-318	0.3231	8.85E-1055	0.4523
f_5	2.5	3	8.24E-355	0.3565	6.37E-353	0.3743	4.94E-356	0.3078	1.33E-1228	0.4510
f_6	1	0.5	2.41E-265	0.3735	9.30E-261	0.3615	9.17E-269	0.2918	2.29E-975	0.4269
f_7	0.5	1	2.05E-133	0.3573	2.38E-133	0.3627	4.27E-134	0.3115	1.19E-504	0.4287

6. Conclusions

The current study, we have proposed a general two-step iterative method for solving nonlinear equations. An analytic proof of convergence order of this method demonstrates that the method has an optimal convergence order four. This method requires only three function evaluations in per full step, so the EI is $\sqrt[3]{4} = 1.587$, which is optimal in the sense of Kung and Traub. Finally, by using the previous iterations of the method, we have extended it to the method with memory and increasing the EI to $\sqrt[3]{6} = 1.8171$.

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