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An efficient iterative method for finding the Moore-Penrose and Drazin inverse of a matrix

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In this paper, a third order convergent method for finding the Moore-Penrose inverse of a matrix is presented and analysed. Then, we develop the method to find Drazin inversion. This method is very robust to find the Moore-Penrose and Drazin inverse of a matrix. Finally, numerical examples show that the efficiency of the proposed method is superior over other proposed methods. Copyright © 2022 Shahid Beheshti University.

Keywords: Moore-Penrose inverse; Iterative method; Third-order convergence.

1. Introduction

Computing the inverse matrix, especially at high sizes, has always been a very difficult and time consuming task. Therefore, numerical methods have always been important for calculation of the inverse matrix, and when it comes to numerical methods, iterative methods have a special place among these methods.

The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger \in \mathbb{C}^{m\times n}$, is a unique matrix X satisfying in [13]:

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA, \tag{1}$$

where above equations are Penrose equations, and where A^* is the conjugate transpose of A.

There are two direct methods for finding the Moore-Penrose inverse, that means by their singular value decomposition and determinantal representations [15, 17, 6, 7]. In the following, we present a basic method for finding the Moore-Penrose inverse based on the singular value decomposition [19].

Assume that $rank(A) = s$. There are orthogonal matrices $U \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{n\times n}$, such that

$$A = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} W^*,$$

where

$$D = diag(\sigma_1, \dots, \sigma_s), \quad \sigma_1 \geq \cdots \geq \sigma_s > 0,$$

and $\sigma_1^2, \dots, \sigma_s^2$, are the nonzero eigenvalues of A^*A. Therefore, the Moore-Penrose inverse $A^\dagger$ could be defined by

$$A^\dagger = W \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^*.$$

There are several iterative methods to compute the Moore-Penrose inverse. The most famous iterative methods for approximating the inverse A^{-1} are the Newton's and Chebyshev's methods [5, 8]

$$V_{r+1} = V_r(2I - AV_r), \quad r = 0, 1, 2, \cdots, \tag{2}$$

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and

$$\begin{cases} W_r = AV_r, \\ V_{r+1} = V_r\big(3I - W_r(3I - W_r)\big), \end{cases} \tag{3}$$

respectively, where I is the identity matrix.

In [3], proposed another second order method as follows:

$$\begin{cases} W_r = AV_r, \\ V_{r+1} = V_r\big(5.5I - W_r(8I - 3.5W_r)\big), \end{cases} \tag{4}$$

and this method is better in terms of efficiency and computation than all the proposed methods.

Note that an initial matrix was developed and introduced in [12] as follows

$$V_0 = \alpha A^*, \tag{5}$$

here $\alpha = \dfrac{1}{\|A\|_1\|A\|_\infty}$, where

$$\|A\|_1 = \max_{1\le j\le n}\sum_{i=1}^{m}|a_{ij}|, \qquad \|A\|_\infty = \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|.$$

In the following, we show $\|X\|$, for any subordinate norm of matrix X.

2. An efficient iterative method

Assume that $A \in \mathbb{C}^{n\times n}$ is the nonsingular marix. We propose iterative method to find A^{-1}:

$$V_{r+1} = \frac{1}{25}V_r\Big(225I - 669AV_r + 907(AV_r)^2 - 582(AV_r)^3 + 144(AV_r)^4\Big), \tag{6}$$

or

$$\begin{cases} \vartheta_r = AV_r, \\ \xi_r = \vartheta_r^2, \\ V_{r+1} = \dfrac{1}{25}V_r\Big(225I - 669\vartheta_r + \xi_r\big(907I - 582\vartheta_r + 144\xi_r\big)\Big). \end{cases} \tag{7}$$

In the following, we give some properties about the iterative method (7).

Theorem 1 *Let $A \in \mathbb{C}^{n\times n}$ be a nonsingular matrix, and the initial approximation V_0 satisfies in*

$$\|I - AV_0\| < 1,$$

then the iterative method (7) converges to A^{-1} with third-order.

Proof. Consider $E_r = I - AV_r$, therefore

$$\begin{aligned} E_{r+1} &= I - AV_{r+1} = (I - AV_r)^3\Big(I - 6AV_r + \frac{144}{25}(AV_r)^2\Big) \\ &= (I - AV_r)^3\Big(\frac{19}{25}I - \frac{138}{25}(I - AV_r) + \frac{144}{25}(I - AV_r)^2\Big) = \frac{19}{25}E_r^3 - \frac{138}{25}E_r^4 + \frac{144}{25}E_r^5. \end{aligned} \tag{8}$$

So,

$$\|E_{r+1}\| = \|\frac{19}{25}E_r^3 - \frac{138}{25}E_r^4 + \frac{144}{25}E_r^5\| \le \frac{19}{25}\|E_r^3\| + \frac{138}{25}\|E_r^4\| + \frac{144}{25}\|E_r^5\|. \tag{9}$$

Therefore, we have $E_r \to 0$, and then $\|E_0\| < 1$, so $I - AV_r \to 0$, it means $X_r \to A^{-1}$.

Now, suppose that $\epsilon = A^{-1} - V_r$ is the error matrix, then we can get $A\epsilon_r = I - AV_r = E_r$ and

$$A\epsilon_{r+1} = \frac{19}{25}(A\epsilon_r)^3 - \frac{138}{25}(A\epsilon_r)^4 + \frac{144}{25}(A\epsilon_r)^5.$$

Therefore, since A is invertible, we obtain

$$\epsilon_{r+1} = \frac{19}{25}\epsilon_r(A\epsilon_r)^2 - \frac{138}{25}\epsilon_r(A\epsilon_r)^3 + \frac{144}{25}\epsilon_r(A\epsilon_r)^4$$

Consequently, we have

$$\|\epsilon_{r+1}\| \le \Big(\frac{19}{25}\|A\|^2 + \frac{138}{25}\|A\|^3\|\epsilon_r\| + \frac{144}{25}\|A\|^4\|\epsilon_r\|^2\Big)\|\epsilon_r\|^3.$$

Here, the iterative method (7) converges to A^{-1}, and the order convergence of the method is three. □

Theorem 2 *According to (5), iterative method (7) satisfies*

$$\lim_{r\to\infty} \frac{t_{r+1}}{t_r^3} = \lim_{r\to\infty} \frac{s_{r+1}}{s_r^3} = \frac{19}{25},$$

where $t_r = \|A\epsilon_r\|$ *and* $s_r = \|E_r\|$.

Proof. From Theorem 1, we have

$$A\epsilon_{r+1} = \frac{19}{25}(A\epsilon_r)^3 - \frac{138}{25}(A\epsilon_r)^4 + \frac{144}{25}(A\epsilon_r)^5,$$

then, we can obtain

$$t_{r+1} = \|A\epsilon_{r+1}\| \geq \frac{19}{25}\|A\epsilon_r\|^3 - \frac{138}{25}\|A\epsilon_r\|^4 + \frac{144}{25}\|A\epsilon_r\|^5 = t_r^3(\frac{3}{4} - \frac{23}{4}t_r + 6t_r^2).$$

On the other hand, we can get

$$t_{r+1} = \|A\epsilon_{r+1}\| \leq \frac{19}{25}\|A\epsilon_r\|^3 + \frac{138}{25}\|A\epsilon_r\|^4 + \frac{144}{25}\|A\epsilon_r\|^5 = t_r^3(\frac{19}{25} + \frac{138}{25}t_r + \frac{144}{25}t_r^2).$$

Now, from last two inequalities, we have

$$\frac{19}{25} + \frac{138}{25}t_r + \frac{144}{25}t_r^2 \leq \frac{t_{r+1}}{t_r^3} \leq \frac{19}{29} + \frac{138}{25}t_r + \frac{144}{25}t_r^2.$$

According to Theorem 1 and $\lim_{r\to\infty} t_r = 0$, then we can conclude that

$$\lim_{r\to\infty} \frac{t_{r+1}}{t_r^3} = \frac{19}{25}.$$

In the following, from Theorem 1, we have

$$E_{r+1} = \frac{19}{25}E_r^3 - \frac{138}{25}E_r^4 + \frac{144}{25}E_r^5,$$

like previous inequalities

$$\frac{19}{25} + \frac{138}{25}s_r + \frac{144}{25}s_r^2 \leq \frac{s_{r+1}}{s_r^3} \leq \frac{19}{25} + \frac{138}{25}s_r + \frac{144}{25}s_r^2.$$

Now from $\lim_{r\to\infty} s_r = 0$, we can get

$$\lim_{r\to\infty} \frac{s_{r+1}}{s_r^3} = \frac{19}{25}.$$

□

Theorem 3 *Suppose that* A *is a nonsingular matrix. If* $AV_0 = V_0A$ *, then for the sequence (7), we have*

$$AV_i = V_iA, \quad i = 1, 2, \cdots.$$

Proof. At frist, by using $AV_0 = V_0A$ and (7), we can get $AV_1 = V_1A$.
Now, we suppose that $AV_i = V_iA$ is true, then from (7), we can obtain

$$\begin{aligned} AV_{i+1} &= \frac{1}{25}AV_i\Big(225I - 669AV_i + 907(AV_i)^2 - 582(AV_i)^3 + 144(AV_i)^4\Big) \\ &= \frac{1}{25}V_iA\Big(225I - 669V_iA + 907(V_iA)^2 - 582(V_iA)^3 + 144(V_iA)^4\Big) \\ &= V_{i+1}A. \end{aligned}$$

The proof is complete. □

Lemma 1 *For the sequence* $\{V_i\}_{i=0}^{i=\infty}$ *generated by the iterative scheme (7), it holds that*

$$(V_iA)^* = V_iA, \quad (AV_i)^* = AV_i, \quad V_iAA^\dagger = V_i, \quad A^\dagger AV_i = V_i. \tag{10}$$

Proof. We can prove using the principle of mathematical induction on k. For $k=0$ and according to (5), the first two equations are obvious. Using $(AA^\dagger)^* = AA^\dagger$ and $(A^\dagger A)^* = A^\dagger A$, we have

$$V_0AA^\dagger = \alpha A^*AA^\dagger = \alpha A^* = V_0, \qquad A^\dagger AV_0 = \alpha A^\dagger AA^* = \alpha A^* = V_0.$$

Now, suppose that the conclusion holds for some $k>0$. We domenstrate that it hold for $k+1$. So

$$(V_{k+1}A)^* = \{\frac{1}{25}V_k\Big(225I - 669AV_k + 907(AV_k)^2 - 582(AV_k)^3 + 144(AV_k)^4\Big)A\} = V_{k+1}A,$$

and the second statement is proved in a similar way.

For the third statement in (10), using $V_kAA^\dagger = V_k$, or then $AV_kAA^\dagger = AV_k$, we have

$$\begin{aligned} V_{k+1}AA^\dagger &= \frac{1}{25}V_k\Big(225I - 669AV_k + 907(AV_k)^2 - 582(AV_k)^3 + 144(AV_k)^4\Big)AA^\dagger \\ &= \frac{1}{25}V_k\Big(225I - 669AV_k + 907(AV_k)^2 - 582(AV_k)^3 + 144(AV_k)^4\Big) = V_{k+1}. \end{aligned}$$

Therefore, the third statement in (10) holds for $k+1$. Finally, the fourth statement is proved in a similar way. □

Theorem 4 *Considering the same assumptions as in Theorem 1, the iterative method (7) is asymptotically stable.*

Proof. This theorem is similar to those have been taken for a general family of methods in [4]. Thus, the proof is omitted. □

Lemma 2 *[4] For $M \in \mathbb{C}^{n\times n}$ and any given $\xi > 0$. There is at least one matrix norm $\|.\|$such that:*

$$\rho(M) \le \|M\| \le \rho(M) + \xi,$$

here $\rho(M) = max|\lambda_i|$, where λ_i are eigenvalue of matrix M.

Lemma 3 *[16] For $P, S \in \mathbb{C}^{n\times n}$, such that $P = P^2$ and $PS = SP$, it holds that*

$$\rho(PS) \le \rho(S).$$

Theorem 5 *Assume that $A \in \mathbb{C}^{n\times m}$ is a matrix of rank s, and $\sigma_1 > \sigma_2 > \cdots > \sigma_s > 0$ are singular values of A. Then the generated sequence of (7) converges to the Moore-Penrose inverse $A^\dagger$ in thirth-order provided that $X_0 = \frac{A^*}{C}$, where $C > \sigma_1^2$ is a constant.*

Proof. According to Lemma 1, we have

$$\|X_{r+1} - A^\dagger\| = \|V_{r+1}AA^\dagger - A^\dagger AA^\dagger\| \le \|V_{r+1}A - A^\dagger A\|\|A^\dagger\|, \tag{11}$$

and if $E_r = V_r - A^\dagger$, then $A^\dagger AE_rA = E_rA$.

We claim that

$$E_{r+1}A = \Big(\frac{19}{25}(E_rA)^3) + \frac{138}{25}(E_rA)^4 + \frac{144}{25}(E_rA)^5\Big)(E_rA)^3. \tag{12}$$

We know, from the definitions of the Moore-Penrose inverse and E_r, we have

$$(I - A^\dagger A)^t = I - A^\dagger A, \quad t = 2,3, \qquad (I - A^\dagger A)E_rA = 0, \qquad E_rA(I - A^\dagger A) = 0.$$

By using iterative method (7), we can get

$$\begin{aligned} E_{r+1}A &= \Big[\frac{1}{25}V_r\Big(225I - 669AV_r + 907151(AV_r)^2 - 582(AV_r)^3 + 144(AV_r)^4\Big) - A^\dagger\Big]A \\ &= -(I - V_rA)^3\Big(\frac{19}{25}I - \frac{138}{25}(I - V_rA) + \frac{144}{25}(I - V_rA)^2\Big) + I - A^\dagger A \\ &= -(I - A^\dagger A - E_rA)^3\Big(\frac{19}{25}I - \frac{138}{25}(I - A^\dagger A - E_rA) + \frac{144}{25}(I - A^\dagger A - E_rA)^2\Big) + I - A^\dagger A \\ &= -\Big((I - A^\dagger A) - 3(I - A^\dagger A)E_rA + 3(I - A^\dagger A)(E_rA) - (E_rA)^3\Big) \\ &\quad \times \Big(\frac{19}{25}I - \frac{138}{25}(I - A^\dagger A) + \frac{138}{25}(E_rA) + \frac{144}{25}(I - A^\dagger A) \\ &\quad - \frac{288}{25}(I - A^\dagger A)(E_rA) + \frac{144}{25}(E_rA)^2\Big) + I - A^\dagger A. \end{aligned}$$

Therefore, we can obtain

$$E_{r+1}A = \Big(\frac{19}{25}(E_rA)^3\Big) + \frac{138}{25}(E_rA)^4 + \frac{144}{25}(E_rA)^5\Big)(E_rA)^3.$$

This completes the proof of (12).

Now, consider $P = A^\dagger A$ and $S = X_0A - I$, therefore we have $P^2 = P$ and then

$$\begin{aligned} PS &= A^\dagger A(X_0 - I) = A^\dagger AX_0A - A^\dagger A = (A^\dagger A)^* X_0A - A^\dagger A \\ &= X_0A - A^\dagger A = X_0AA^\dagger A - A^\dagger A = (X_0A - I)A^\dagger A = SP. \end{aligned}$$

Now, according Lemma 3, we can get

$$\rho((X_0 - A^\dagger)A) = \rho\Big(\big(\frac{A^*}{C} - A^\dagger\big)A\Big) \le \rho\Big(\frac{A^*}{C}A - I\Big) = \max_{1\le i\le s}\Big|1 - \lambda_i\Big(\frac{A^*}{C}A\Big)\Big|.$$

Since $C < \sigma_1^2$, then we have

$$\max_{1\le i\le s}\Big|1 - \lambda_i\Big(\frac{A^*}{C}A\Big)\Big| < 1. \tag{13}$$

By applying Lemma 2, we can conclude

$$\Big\|(X_0 - A^\dagger)A\Big\| \le \rho\Big((X_0 - A^\dagger)A\Big) + \xi < 1.$$

According to (11) and (12), we have $\lim_{r\to\infty}\|X_r - A^\dagger\| = 0$. □

Theorem 6 *Sequence V_r produced by (7) and with (5), satisfies*

$$\mathcal{R}(V_r) = \mathcal{R}(A^*), \qquad \mathcal{N}(V_r) = \mathcal{N}(A^*),$$

for $r \ge 0$, where $\mathcal{R}(.)$ denotes the range of matrix and $\mathcal{N}(.)$ is the null space of matrix.

Proof. At frist since $V_0 = \alpha A^*$, then the theorem obviously holds for $r = 0$. Suppose that $y \in \mathcal{N}(V_r)$ is arbitrary vector. According to the method (7), we have

$$V_{r+1}y = \frac{1}{25}\Big(225V_ry - 669V_rAV_ry + 907V_r(AV_r)^2y - 582(V_rAV_r)^3y + 144V_r(AV_r)^4y\Big) = 0.$$

So $y \in \mathcal{N}(V_{r+1})$, then we can conclude $\mathcal{N}(V_r) \subseteq \mathcal{N}(V_{r+1})$. Similarly we have $\mathcal{R}(V_r) \supseteq \mathcal{R}(V_{r+1})$. Therefore, by mathematical induction we have

$$\mathcal{N}(V_r) \supseteq \mathcal{N}(V_0) = \mathcal{N}(A^*), \qquad \mathcal{R}(V_r) \subseteq \mathcal{R}(V_0) = \mathcal{R}(A^*).$$

To prove equality, let

$$\mathcal{N} = \bigcup_{r\in\mathbb{N}_0}\mathcal{N}(V_r).$$

Suppose that $y \in \mathcal{N}$, then $y \in \mathcal{N}(V_{r_0})$ for $r_0 \in \mathbb{N}_0$. Since $y \in \mathcal{N}(V_r)$ for every $r \ge r_0$, then we have $V_ry = 0$ and according Theorem 1, we obtain

$$Vy = \lim_{r\to+\infty} V_ry = 0.$$

Finally $y \in \mathcal{N}(V) = \mathcal{N}(A^*)$ and $\mathcal{N} \subseteq \mathcal{N}(A^*)$. On the other hand, by using

$$\mathcal{N}(A^*) \subseteq \mathcal{N}(V_r) \subseteq \mathcal{N} \subseteq \mathcal{N}(A^*),$$

we have, $\mathcal{N}(V_r) = \mathcal{N}(A^*)$.

Now, according to relation

$$dim\ \ \mathcal{R}(V_r) = m - dim\ \ \mathcal{N}(V_r) = m - dim\ \ \mathcal{N}(A^*) = dim\ \ \mathcal{R}(A^*),$$

and $\mathcal{R}(V_r) \subseteq \mathcal{R}(A^*)$, we have $\mathcal{R}(V_r) = \mathcal{R}(A^*)$. □

Now, if $rank(A) = s \le min\{n, m\}$, the singular value decomposition of A is:

$$A = U\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}W^*.$$

Suppose that $V_0 = \alpha A^*$, and from iterative method (7), we have

$$V_0 = \alpha A^* = W\begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix}U^*,$$

where $D_0 = \alpha D$ is a diagonal matrix. Therefore, we have

$$W^* V_0 U = \begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, from the principle of mathematical induction and Lemma 1, we have

$$W^* V_r U = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}, \tag{14}$$

where D_r is diagonal matrix:

$$D_{r+1} = \frac{1}{25} D_r \Big(225I - 669(DD_r) + 907(DD_r)^2 - 582(DD_r)^3 + 144(DD_r)^4 \Big), \tag{15}$$

In the following, we state the following theorem.

Theorem 7 *Suppose that $A \in \mathbb{C}^{n\times m}$ is a matrix of rank s. The sequence (7) with the initial matrix (5), relations (14) and (15) hold.*

Table 1. CO and MM in every iteration of various methods

Method	Newton	Chebyshev	Method 1	Method 2
CO	2	3	2	3
MM	2	3	3	4

Table 1 denotes convergence order (CO) and the number of matrix-matrix multiplications (MM) in every iteration of various methods.

Note that Newton's method is shown with $Newton$ and methods (3), (4) and (7) with $Chebyshev$, $Method$ 1 and $Method$ 2, respectively.

3. Application in Finding the Drazin Inverse

In 1958, a different kind of generalized inverse was introduced by Drazin [2], which does not have flexibility in rings and semi-groups of associations but commutes with the element. The importance of this type of inverse and its calculation was later expressed by Wilkinson in [22]. Many authors then offer direct or iterative methods for calculating this problem [14, 11, 21, 10].

Definition 1 *The smallest negative integer $k = ind(A)$, that holds*

$$rank(A^{k+1}) = rank(A^k),$$

is called index of matrix A.

Definition 2 *Suppose that $A \in \mathbb{C}^{n\times n}$, the Drazin inverse of A, denoted by A^D, is matrix V, which holds in the following equations:*

$$A^k V A = A^k, \qquad VAV = V, \qquad AV = VA,$$

where $k = ind(A)$.

- If $ind(A) = 0$, then $A^D = A^{-1}$.
- If $ind(A) = 1$, then $A^D = A^{\dagger}$.

Li and Wei [9] proved that method (2) can be used to find the Drazin inverse of square matrices. Here, they proposed the initial matrix:

$$V_0 = W_0 = \beta A^l, \qquad l \geq ind(A) = k, \tag{16}$$

Where β must satisfy the condition $\|I - AV_0\| < 1$.

Now, we present the iterative method (7) for finding the Drazin inverse, where the initial matrix as

$$V_0 = W_0 = \frac{2}{tr(A^{k+1})} A^k, \tag{17}$$

where $tr(.)$ stands for the trace of matrix A.

Proposition 1 *[20] Let $P_{L,M}$ be the projector on a space L along a space M, then*

(i) $P_{L,M}Q = Q \Leftrightarrow \mathcal{R}(Q) \subseteq L$;

(ii) $QP_{L,M}Q = Q \Leftrightarrow \mathcal{N}(Q) \supseteq M$,

where $\mathcal{R}(.)$ denotes the range of matrix and $\mathcal{N}(.)$ is the null space of matrix.

Theorem 8 *Suppose that A is singular square matrix. Also, let the initial matrix is chosen by equation (17). Then the sequence W_r generated by the iterative method (7) has an error as follows*

$$\|A^D - W_r\| \leq \mathcal{O}(\|A^D\|\|F_0\|^{3^r}),$$

to find the Drazin inverse, where $F_0 = T - AW_0$.

Proof. Let $F_0 = I - AW_0$, then $F_r = I - AW_r$. Thus similarly as in (8), we can get

$$\begin{aligned} F_{r+1} &= I - AW_{r+1} \\ &= (I - AW_r)^3\Big(\frac{19}{25}I - \frac{138}{25}(I - AW_r) + \frac{144}{25}(I - AW_r)^2\Big) \qquad (18) \\ &= \frac{19}{25}F_r^3 - \frac{138}{25}F_r^4 + \frac{144}{25}F_r^5. \end{aligned}$$

Using an arbitrary matrix norm of (18), we have

$$\|F_{r+1}\| \leq \frac{19}{25}\|F_r\|^3 + \frac{138}{25}\|F_r\|^4 + \frac{144}{25}\|F_r\|^5. \qquad (19)$$

Here, since $\|F_0\| < 1$, from relation (19), we can obtain

$$\|F_1\| \leq \frac{19}{25}\|F_0\|^3 + \frac{138}{25}\|F_0\|^4 + \frac{144}{25}\|F_0\|^5 \leq \mathcal{O}(\|F_0\|^3).$$

By continuing this process, we can conclude

$$\|F_{r+1}\| \leq \frac{19}{25}\|F_r\|^3 + \frac{138}{25}\|F_r\|^4 + \frac{144}{25}\|F_r\|^5 \leq \mathcal{O}(\|F_r\|^3).$$

So, we have $\|F_{r+1}\| \leq \mathcal{O}(\|F_r)\|$, for every $r \geq 0$. Thus, we can get

$$\|F_r\|^3 \leq \mathcal{O}(\|F_0\|^{3^r}), \qquad r \geq 0. \qquad (20)$$

According to relation (16), we have $\mathcal{R}(W_0) \subseteq \mathcal{R}(A^k)$. In addition, the use of this fact together with (6), which implies that $\mathcal{R}(W_r) \subseteq \mathcal{R}(W_{r-1})$, so, we can get

$$\mathcal{R}(W_r) \subseteq \mathcal{R}(A^k), \qquad r \geq 0. \qquad (21)$$

From (6), we have

$$W_{r+1} = \frac{1}{25}W_r\Big(225I - 669AW_r + 907(AW_r)^2 - 582(AW_r)^3 + 144(AW_r)^4\Big),$$

it is easy to verify that

$$\mathcal{N}(W_r) \supseteq \mathcal{N}(A^k), \qquad r \geq 0. \qquad (22)$$

Now, using from [1], we have

$$AA^D = A^DA = P_{\mathcal{R}(A^k),\mathcal{N}(A^k)}, \qquad (23)$$

By usin Proposition 1 and relations (21) and (22), we obtain

$$W_rAA^D = W_r = A^DAW_r, \qquad r \geq 0. \qquad (24)$$

Therefore, if the error matrix is $\delta_r = A^D - W_r$, then

$$\delta_r = A^D - W_r = A^D - A^DAW_r = A^D(I - AW_r) = A^DF_r. \qquad (25)$$

From (25) and (20), we have

$$\|\delta_r\| = \|A^D\|\|F_r\| \leq \mathcal{O}(\|A^D\|\|F_0^{3^r}\|),$$

the proof is complete. □

Corollary 1 *According to the condition of theorem 8 and the initial iteration W_0 is chosen such that*

$$\|F_0\| \leq \|I - AW_0\| < 1, \qquad (26)$$

the scheme (7) converges to A^D.

4. Numerical results

In this section, we review and compare the results of the proposed method with other methods. Since the comparison of Method 1 in [3] shows that it worked better than other methods, so we compare the proposed Method 2 with methods Newton, Chebyshev and Method 1. The stop criterion is

$$\frac{\|V_{r+1} - V_r\|_\infty}{1 + \|V_r\|_\infty} < 10^{-12}.$$

Here, Res is the residual when the process is stopped, and CPU is the time spent.

The algorithm to find A^{-1} by using method (7) is the following.

Algorithm 1 The method (7) for computing invers matrix.

Step 1: Input matrix $A \in \mathbb{C}^{n\times n}$.

Step 2: Take initial matrices $V_0 = \dfrac{A^*}{\|A\|_1\|A\|_\infty}$ and tolerance $\varepsilon \geq 0$. Set $r := 0$.

Step 3: Let

$$\vartheta_r = AV_r,$$

$$\xi_r = \vartheta_r^2,$$

$$V_{r+1} = \frac{1}{25}V_r\Big(225I - 669\vartheta_r + \xi_r\big(907I - 582\vartheta_r + 144\xi_r\big)\Big).$$

Step 4: Stop if $\dfrac{\|V_{r+1} - V_r\|_\infty}{1 + \|V_r\|_\infty} \leq \varepsilon$. Otherwise, $r := r + 1$ go to Step 3.

Example 1 *Consider real three diagonal matrix with 100 arrays, where diagonals are:*

$$(1, 20) = 6, \qquad (1, 1) = 1.5, \qquad (50, 1) = -4.5.$$

The results of Example 1 are displayed in Figures 1 and 2.

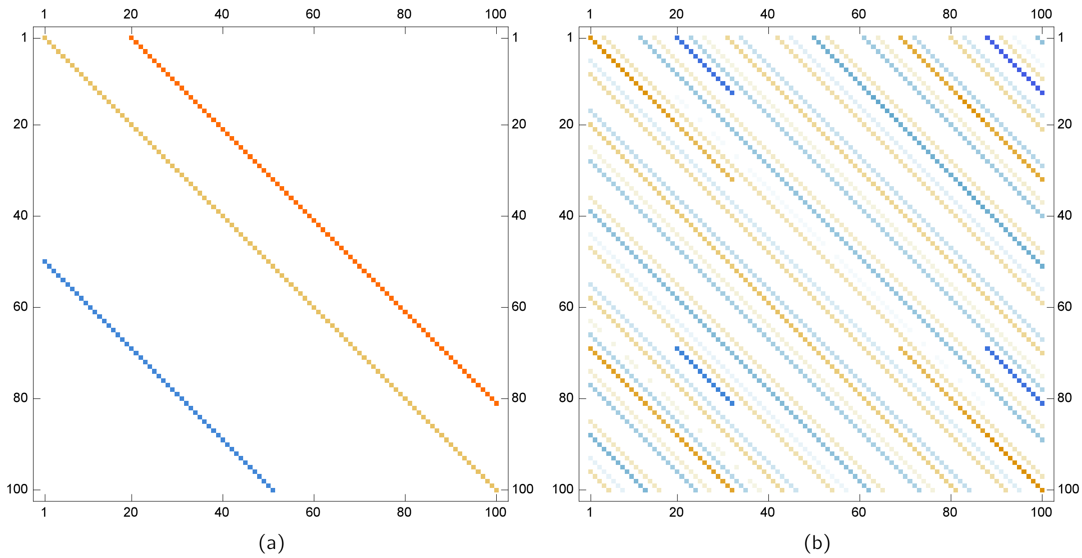


Figure 1. a, b, respectively represent plots matrix and inverse matrix Example 1.

Example 2 *Consider complex three diagonal matrix with 100 arrays, where diagonals are:*

$$(1, 50) = 9 - 2i, \qquad (1, 1) = 1 + 0.5i, \qquad (2, 1) = 4 - i.$$

The results of Example 2 are showed in Figures 3 and 4.

Example 3 *To evaluate the efficiency of the method in general, we have studied several real random matrices with different dimensions. The results are presented in Figure 5.*

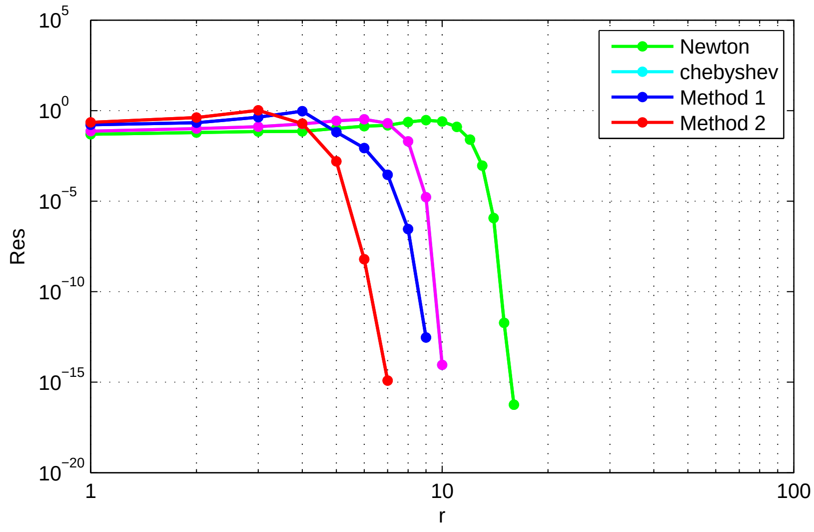
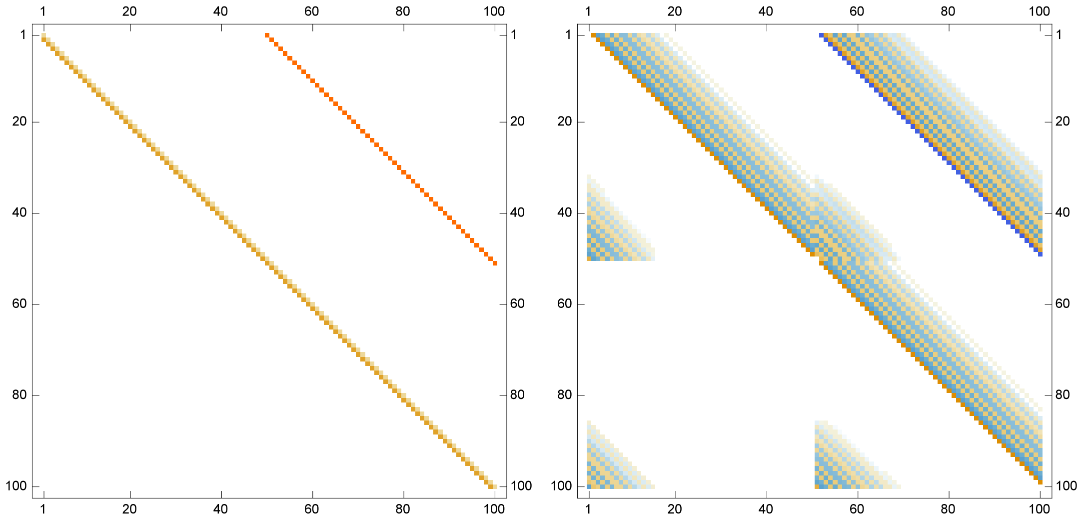
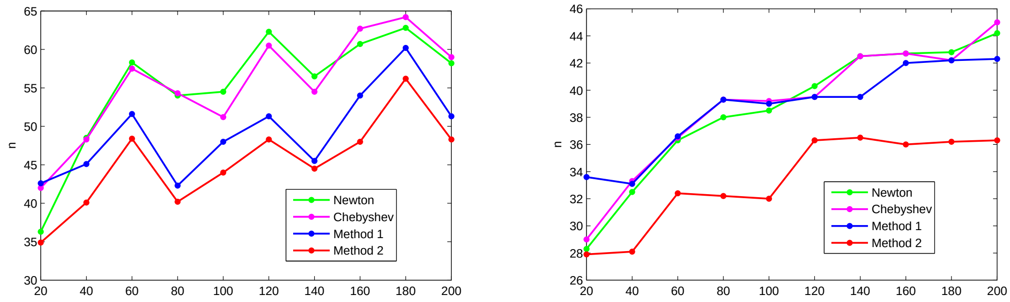


Figure 2. The numerical results of the methods for Example 1.



(a) (b)

Figure 3. a, b, respectively show plots matrix and inverse matrix Example 2.



(a) (b)

Figure 5. a ,b, respectively, represent the average MM to compute the Moore- Penrose inverse of real matrices $(n \times n)$ and $(n \times n+20)$ by different methods.

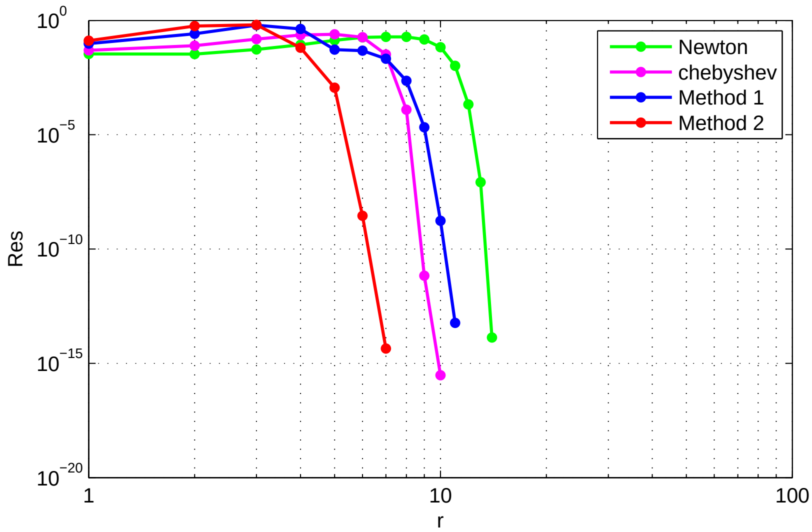


Figure 4. The numerical results of the methods for Example 2.

Example 4 *[18] Consider three diagonal matrix, where diagonals are:*

$$(1,2) = 1, \qquad (1,1) = 0, \qquad (2,1) = -1.$$

Here dimension is odd, and matrices are singular with $ind(A) = 1$. The results for conputing Drasin inverse matrices are presented in Figures 6 and 7.

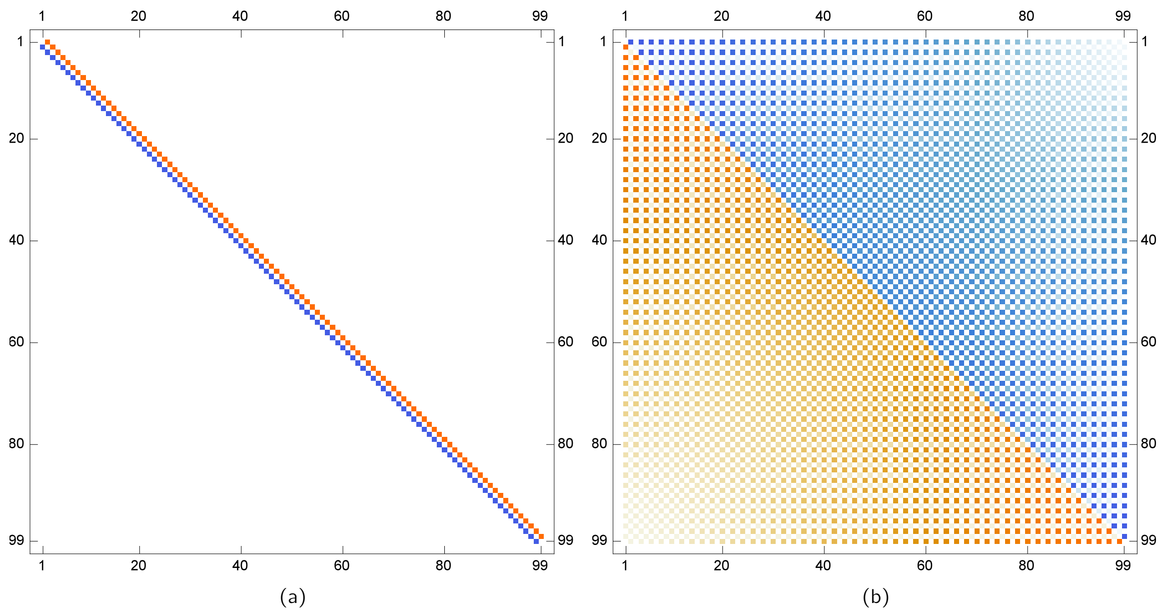


Figure 6. a, b, respectively display plots matrix and Drasin inverse matrix Example 4 for $n = 99$.

5. Conclusions

In this paper, we first present a method for solving nonlinear equations. This method has a very low performance in terms of computational efficiency due to the presence of consecutive derivatives. Then we try the same method to find the Moore-Penrose and Drazin inverse of an arbitrary matrix. We discuss the details of the performance of the iterative method in this study:

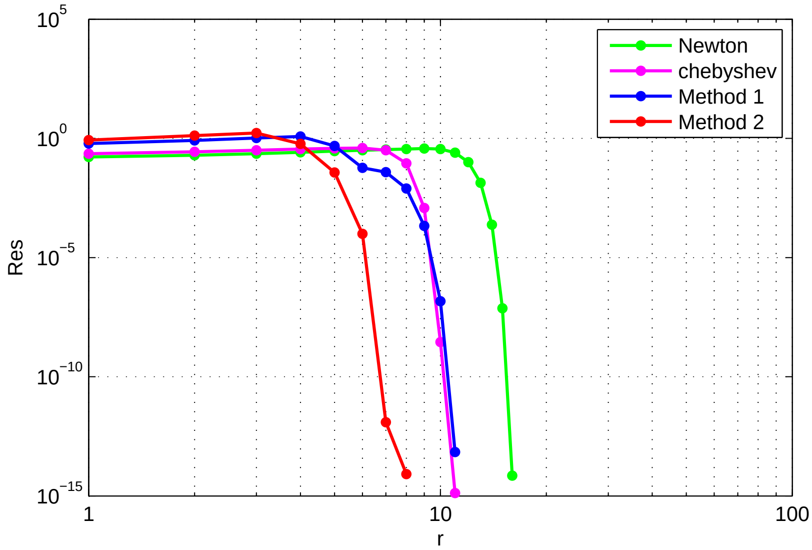


Figure 7. The numerical results of the methods for Example 4.

- Demonstrate desired results by providing specific real and complex examples.
- Provide optimal results for real square and rectangular random matrices in different dimensions (see Figures 5).
- Provide examples in different dimensions to compute the Drazin inverse with the developed iterative method.

The mentioned results show that the proposed method well covered by theory.

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