Research Article Computational Mathematics and Computer Modeling with Applications

Received 15 February 2022
Accepted 18 March 2022

DOI: 10.52547/CMCMA.1.1.63
AMS Subject Classification: 49M15; 47H17; 65J15; 65G99; 41A25

On the semi-local convergence of the Homeier method in Banach space for solving equations

Samundra Regmi[a], Ioannis K. Argyros[b] Santhosh George[c] and Christopher I. Argyros[d]

In this paper we consider the semi-local convergence analysis of the Homeier method for solving nonlinear equation in Banach space. As far as we know no semi-local convergence has been given for the Homeier under Lipschitz conditions. Our goal is to extend the applicability of the Homeier method in the semi-local convergence under these conditions. We use majorizing sequences and conditions only on the first derivative which appear on the method for proving our results. Numerical experiments are provided in this study.

Keywords: semi-local convergence; Homeier method; iterative methods; Banach space; convergence criterion.

1. Introduction

We consider the nonlinear equation

$$\mathcal{G}(x) = 0, \tag{1.1}$$

where $\mathcal{G} : \Omega \subset B \longrightarrow B_1$ is an operator acting between Banach spaces B and B_1 with $\Omega \neq \emptyset$. In general a closed form solution for (1.1) is not possible, so iterative methods are used for approximating a solution x_* of (1.1). Many iterative methods are studied for approximating x_* [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

In this study, we consider the Homeier method, defined for $n = 0, 1, 2, \ldots,$ by

$$\begin{aligned} y_n &= x_n - \frac{1}{2}\mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n), \\ x_{n+1} &= x_n - \mathcal{G}'(y_n)^{-1}\mathcal{G}(x_n). \end{aligned} \tag{1.2}$$

The local convergence of the Homeier method was shown to be of order three using Taylor expansion and assumptions on the fourth order derivative of $\mathcal{G}$, which is not on these methods [13]. So, the assumptions on the fourth derivative reduce the applicability of these methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

For example: Let $B = B_1 = \mathbb{R}$, $\Omega = [-0.5, 1.5]$. Define λ on Ω by

$$\lambda(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0, \\ 0 & if\ t = 0. \end{cases}$$

Then, we get $t^* = 1$, and

$$\lambda'''(t) = 6 \log t^2 + 60t^2 - 24t + 22.$$

[a] Learning Commons, University of North Texas at Dallas, Dallas, TX, USA. Email: samundra.regmi@untdallas.edu.
[b] Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA. Email: iargyros@cameron.edu.
[c] Department of Mathematical and Computational Sciences,National Institute of Technology Karnataka, India-575 025. Email: sgeorge@nitk.edu.in.
[d] Department of Computing and Technology, Cameron University, Lawton, OK 73505, USA. Email: christopher.argyros@cameron.edu.
* Correspondence to: I.K. Argyros.

Obviously $\lambda'''(t)$ is not bounded on Ω. So, the convergence of method (1.2) is not guaranteed by the previous analyses in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27].

In this study we introduce a majorant sequence and use general continuity conditions to extend the applicability of Homeier method. Our analysis includes error bounds and results on uniqueness of x_* based on computable Lipschitz constants not given before in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27] and in other similar studies using Taylor series. Our idea is very general. So, it applies on other methods too.

The rest of the study is set up as follows: In Section 2 we present results on majorizing sequences. Sections 3,4 contain the semi-local and local convergence, respectively, where in Section 4 the numerical experiments are presented. Concluding remarks are given in the last Section 5.

2. Majorizing Sequences

Let $\ell_0 > 0, \ell > 0,\ \eta > 0$ be given parameters with $\ell_0 \leq \ell$. Consider sequences $\{t_n\}, \{s_n\}$ as

$$\begin{aligned} t_0 &= 0,\ s_0 = \eta, \\ t_{n+1} &= s_n + \frac{2\bar{\ell}(1+\ell_0 t_n)}{(1-\ell_0 t_n)(1-\ell_0 s_n)}(s_n - t_n)^2 + \frac{(s_n - t_n)(1+\ell_0 t_n)}{1-\ell_0 s_n}, \\ s_{n+1} &= t_{n+1} + \frac{\ell(2(s_n - t_n) + (t_{n+1} - t_n))}{4(1-\ell_0 t_{n+1})}(t_{n+1} - t_n), \end{aligned} \tag{2.1}$$

where $\bar{\ell} = \begin{cases} \ell_0, & if\ \ n = 0 \\ \ell, & if\ \ n = 1, 2, \ldots \end{cases}$.

Next, we present convergence results for sequences $\{t_n\}$.

LEMMA 2.1 *Suppose: Sequence $\{t_n\}$ satisfies*

$$t_n \leq s_n < \frac{1}{\ell_0}. \tag{2.2}$$

Then, sequences $\{t_n\}$ converge to its unique least upper bound $t^* \in [0, \frac{1}{\ell_0}]$.

Proof. Sequences $\{t_n\}$ is nondecreasing by (2.2), bounded from above by $\frac{1}{\ell_0}$ and as such it converges to t^*. □

3. Semi-local convergence

The hypotheses (H) shall be used. Suppose:

(H1) There exists $x_0 \in \Omega$ and $\eta \geq 0$ such that $\mathcal{G}'(x_0)^{-1}$ exists and

$$\|\mathcal{G}'(y_0)^{-1}\mathcal{G}(x_0)\| \leq \eta.$$

(H2)

$$\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(v) - \mathcal{G}'(x_0))\| \leq \ell_0\|v - x_0\|$$

for all $v \in \Omega$. Set $\Omega_0 = \Omega \cap U(x_0, \frac{1}{\ell_0})$.

(H3)

$$\|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(w) - \mathcal{G}'(v))\| \leq \ell\|w - v\|$$

for all $v, w \in \Omega_0$.

(H4) Hypotheses of Lemma 2.1 hold.
and

(H5) $U[x_0, t^*] \subset \Omega$.

Next, the semi-local convergence follows.

THEOREM 3.1 *Suppose the hypotheses (H) hold. Then, the following items hold*

$$\{x_n\} \in U(x_0, t^*) \tag{3.1}$$

$$\|x_* - x_n\| \leq t^* - t_n, \tag{3.2}$$

for some $x^* \in U[x_0, t^*]$ and $F(x^*) = 0$.

Proof. Mathematical induction is used to show

$$\|y_k - x_k\| \leq s_k - t_k \tag{3.3}$$

and

$$\|x_{k+1} - y_k\| \leq t_{k+1} - s_k. \tag{3.4}$$

By method (1.2) and (H4) one gets

$$\|y_0 - x_0\| = \|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\| \leq \eta = s_0 - t_0 = s_0 = \eta \leq \frac{\eta}{1-\beta},$$

so (3.3) holds for $n = 0$ and $y_0 \in U(x_0, t^*)$.

Let $z \in U(x_0, t^*)$. Then, using (H2) one obtains

$$\begin{aligned} \|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(z) - \mathcal{G}'(x_0))\| &\leq \ell_0\|z - x_0\| \\ &\leq \ell_0 t^* < 1, \end{aligned} \tag{3.5}$$

so

$$\|\mathcal{G}'(z)^{-1}\mathcal{G}'(x_0)\| \leq \frac{1}{1 - \ell_0\|z - x_0\|} \tag{3.6}$$

follows from the Lemma on invertible linear operators due to Banach [17]. Hence, $\mathcal{G}'^{-1}(y_0) \in L(B_1, B)$ and x_1 is well defined.

$$\begin{aligned} x_{k+1} &= y_k + \frac{1}{2}(\mathcal{G}'^{-1}(x_k) - \mathcal{G}'^{-1}(y_k))\mathcal{G}(x_k) + \frac{1}{2}\mathcal{G}'^{-1}(y_k)\mathcal{G}(x_k), \\ x_{k+1} - y_k &= \frac{1}{2}\mathcal{G}'^{-1}(x_k)(\mathcal{G}'(y_k) - \mathcal{G}'(x_k))\mathcal{G}'^{-1}(y_k)(-2\mathcal{G}(x_k))(y_k - x_k) \\ &\quad -\frac{1}{2}\mathcal{G}'^{-1}(x_k)(-2\mathcal{G}(x_k))(y_k - x_k). \end{aligned} \tag{3.7}$$

So, by (3.6) (for $z = x_0, y_0$), (H2) and the definition of the sequence $\{t_k\}$ one gets

$$\begin{aligned} \|x_{k+1} - y_k\| &\leq \frac{\ell\|y_k - x_k\|^2(1 + \ell_0\|x_k - x_0\|)}{(1 - \ell_0\|x_0 - x_0\|)(1 - \ell_0\|y_k - x_0\|)} + \frac{(1 + \ell_0\|x_k - y_k\|)\|y_k - x_k\|}{1 - \ell_0\|y_k - x_0\|} \\ &\leq \frac{\bar{\ell}(s_k - t_k)^2(1 + \ell_0 t_k)}{(1 - \ell_0 t_k)(1 - \ell_0 s_k)} + \frac{(1 + \ell_0 t_k)(s_k - t_k)}{1 - \ell_0 s_k} = t_{k+1} - s_k. \end{aligned} \tag{3.8}$$

By the second substep of method (1.2)

$$\begin{aligned} \mathcal{G}(x_{k+1}) &\leq \mathcal{G}(x_{k+1}) - \mathcal{G}(y_k) - \mathcal{G}'(y_k)(x_{k+1} - x_k) \\ &= \int_0^1 (\mathcal{G}'(x_k + \theta(x_{k+1} - x_k))d\theta - \mathcal{G}'(y_k))(x_{k+1} - x_k), \end{aligned} \tag{3.9}$$

so

$$\begin{aligned} \|y_{k+1} - x_{k+1}\| &\leq \frac{1}{2}\|(\mathcal{G}'(x_{k+1})^{-1}\mathcal{G}'(x_0)\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_{k+1})\| \\ &\leq \frac{1}{2}\frac{(\ell\|y_k - x_k\| + \frac{\ell}{2}\|x_{k+1} - x_k\|)\|x_{k+1} - x_k\|}{1 - \ell_0\|x_{k+1} - x_0\|} \\ &\leq \frac{1}{2}\frac{(\ell(s_k - t_k) + \frac{\ell}{2}(t_{k+1} - t_k)(t_{k+1} - t_k)}{1 - \ell_0 t_{k+1}} \\ &= s_{k+1} - t_{k+1}, \end{aligned} \tag{3.10}$$

where we also used (3.6) for $z = x_{k+1}$. Hence, sequence $\{x_n\}$ is complete in Banach space B, so $\lim_{n\longrightarrow\infty} x_n = x_* \in U[x_0, t^*]$. Using (3.9) we deduce $\lim_{k\longrightarrow\infty} \|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_{k+1})\| \leq \frac{1}{2}\lim_{k\longrightarrow\infty}(\ell(s_k - t_k) + \frac{\ell}{2}(t_{k+1} - t_k)(t_{k+1} - t_k) = 0$. Hence, by the continuity of $\mathcal{G}$ it follows that $\mathcal{G}(x_*) = 0$.

□

A uniqueness of the solution result is presented.

PROPOSITION 3.2 *Assume:*
(1) x_* is a simple solution of (1.1).
(2) There exists $\tau \geq t^*$ so that

$$\ell_0(t^* + \tau) < 2. \tag{3.11}$$

Set $\Omega_1 = \Omega \cap U[x_*, \tau]$. Then, x_* is the unique solution of equation (1.1) in the domain Ω_1.

Proof. Let $q \in \Omega_1$ with $\mathcal{G}(q) = 0$. Define $M = \int_0^1 \mathcal{G}'(q + \theta(x_* - q))d\theta$. Using (H2) and (3.11) one obtains

$$\|\mathcal{G}'(x_0)^{-1}(M - \mathcal{G}'(x_0))\| \leq \ell_0 \int_0^1 ((1-\theta)\|q - x_0\| + \theta\|x^* - x_0\|)d\theta \leq \frac{\ell_0}{2}(t^* + \tau) < 1,$$

so $q = x_*$, follows from the invertability of M and the identity $M(q - x_*) = \mathcal{G}(q) - \mathcal{G}(x_*) = 0 - 0 = 0$. □

Another convergence result can be given if we rewrite method (1.2) as

$$x_{n+1} = x_n - \mathcal{G}'(x_n - \frac{1}{2}\mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n))^{-1}\mathcal{G}(x_n). \tag{3.12}$$

Define parameter

$$r_0 = \min\{\frac{2 - \ell(\eta + \eta_0) - \ell_0\eta}{2\ell_0}, \frac{2\eta_0 - \ell(\eta + \eta_0)\eta}{2\ell_0\eta_0}\}$$

to be determined later and function δ

$$\delta = \delta(t) = \frac{\ell(\eta + \eta_0)}{2(1 - \ell_0(t + \frac{\eta}{2}))}.$$

Then, we present a second semi-local convergence result for method (1.2).

THEOREM 3.3 *Suppose:*

(a) There exist $x_0 \in \Omega, \eta_0 > 0$ such that $\mathcal{G}'(x_n - \frac{1}{2}\mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n))^{-1} \in L(B_1, B)$ and

$$\|\mathcal{G}'(x_n - \frac{1}{2}\mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n))^{-1}\mathcal{G}(x_0)\| \leq \eta_0.$$

(b) $\ell_0\eta + \ell(\eta + \eta_0) < 2$, $\ell(\eta + \eta_0)\eta < 2\eta_0$.
(c) Condition (H1), (H2) and (H3) hold and
(d) $U[x_0, r_0] \subset \Omega$.

Then, sequence $\{x_n\}$ generated by (3.12) is well defined in $U(x_0, r_0)$, remains in $U(x_0, r_0)$ and converges to a solution $x^* \in U[x_0, r_0]$ of equation $\mathcal{G}(x) = 0$, so that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \frac{\ell(\|x_n - x_{n-1}\| + \|\mathcal{G}'(x_{n-1})^{-1}\mathcal{G}(x_{n-1})\|)\|x_n - x_{n-1}\|}{2(1 - \ell_0(\|x_n - x_0\| + \frac{1}{2}\|\mathcal{G}'(x_{n-1})^{-1}\mathcal{G}(x_n)\|)} \\ &\leq \delta\|x_n - x_{n-1}\| \leq \delta^n\|x_1 - x_0\| \leq \delta^n\eta, \end{aligned} \tag{3.13}$$

and

$$\|x^* - x_n\| \leq \frac{\delta^n\eta}{1 - \delta}. \tag{3.14}$$

Proof. As in the proof of Theorem 3.1 we can write

$$\begin{aligned} \mathcal{G}(x_{k+1}) &\leq \mathcal{G}(x_{k+1}) - \mathcal{G}(x_k) + \mathcal{G}(x_k) \\ &= \int_0^1 (\mathcal{G}'(x_k + \theta(x_{k+1} - x_k))d\theta - \mathcal{G}'(x_k - \frac{1}{2}\mathcal{G}'(x_k)^{-1}\mathcal{G}(x_k)))(x_{k+1} - x_k), \end{aligned}$$

so

$$\|\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_{k+1})\| \leq \frac{\ell}{2}(\|x_{k+1} - x_k\| + \|\mathcal{G}'(x_k)^{-1}\mathcal{G}(x_k)\|)\|x_{k+1} - x_k\|. \tag{3.15}$$

We also have for $x \in U(x_0, r), r \in (0, r_0)$

$$\begin{aligned} \|\mathcal{G}'(x_0)^{-1}(\mathcal{G}'(x - \frac{1}{2}\mathcal{G}'(x)^{-1}\mathcal{G}(x)) - \mathcal{G}'(x_0)) &\leq \ell_0(\|x - x_0\| + \frac{1}{2}\|\mathcal{G}'(x)^{-1}\mathcal{G}(x)\|) \\ &\leq \ell_0(r + \frac{\eta}{2}) \\ &< \ell(r_0 + \frac{\eta}{2}) < 1, \end{aligned}$$

by the choice of r_0 and r, so

$$\|\mathcal{G}'(x)^{-1}\mathcal{G}'(x_0)\| \leq \frac{1}{1 - \ell_0(\|x_k - x_0\| + \frac{\eta}{2})}. \tag{3.16}$$

Hence, by method (3.12), (3.15) and (3.16) we obtain (3.13), where we also used

$$\begin{aligned} \|\mathcal{G}'(x_k)^{-1}\mathcal{G}(x_k)\| &\leq \frac{\ell(\delta\|x_k - x_{k-1}\| + \eta_0)\|x_k - x_{k-1}\|}{2(1-\ell_0\|x_k - x_0\|)} \\ &\leq \frac{\ell(\eta+\eta_0)\eta}{2(1-\ell_0 r)} \leq \eta_0, \end{aligned} \tag{3.17}$$

by (3.16) and the choice of r_0. Hence, estimate (3.13) holds, and sequence $\{x_k\}$ is fundamental. So there exists $x^* \in U[x_0, r_0]$ such that $\lim_{k\longrightarrow\infty} x_k = x^*$. By letting $k \longrightarrow \infty$ in (3.15), we deduce $\mathcal{G}(x^*) = 0$. Moreover, let $m \geq 0$, then by (3.13)

$$\begin{aligned} \|x_{k+m} - x_k\| &\leq \|x_{k+m} - x_{k+m-1}\| + \|x_{k+m-1} - x_{k+m-2} + \ldots + \|x_{k+1} - x_k\| \\ &\leq (\delta^{k+m-1} + \ldots + \delta^k)\eta \\ &= \delta^k \frac{1-\delta^m}{1-\delta}\eta. \end{aligned} \tag{3.18}$$

By letting $m \longrightarrow \infty$ in (3.18) we obtain (3.14).

The uniqueness result for the solution x^* is identical to Proposition 3.2.

□

4. Numerical Experiments

We provide some examples in this section.

EXAMPLE 4.1 *Define function*

$$h(t) = \xi_0 t + \xi_1 + \xi_2 \sin \xi_3 t,\ x_0 = 0,$$

where ξ_j, $j = 0, 1, 2, 3$ are parameters. Choose $\psi_0(t) = L_0 t$ and $\psi(t) = Lt$. Then, clearly for ξ_3 large and ξ_2 small, $\frac{L_0}{L}$ can be small (arbitrarily). Notice that $\frac{L_0}{L} \longrightarrow 0$.

EXAMPLE 4.2 *Let $B = B_1 = C[0, 1]$ and $\Omega = U[0, 1]$. It is well known that the boundary value problem [11].*

$$\varsigma(0) = 0, (1) = 1,$$

$$\varsigma'' = -\varsigma - \sigma\varsigma^2$$

can be given as a Hammerstein-like nonlinear integral equation

$$\varsigma(s) = s + \int_0^1 Q(s,t)(\varsigma^3(t) + \sigma\varsigma^2(t))dt$$

where σ is a parameter. Then, define $\mathcal{G} : \Omega \longrightarrow B_1$ by

$$[\mathcal{G}(x)](s) = x(s) - s - \int_0^1 Q(s,t)(x^3(t) + \sigma x^2(t))dt.$$

Choose $\varsigma_0(s) = s$ and $\Omega = U(\varsigma_0, \rho_0)$. Then, clearly $U(\varsigma_0, \rho_0) \subset U(0, \rho_0 + 1)$, since $\|\varsigma_0\| = 1$. Suppose $2\sigma < 5$. Then, conditions (A) are satisfied for

$$L_0 = \frac{2\sigma + 3\rho_0 + 6}{8},\ L = \frac{\sigma + 6\rho_0 + 3}{4},$$

and $\eta = \frac{1+\sigma}{5-2\sigma}$. Notice that $L_0 < L$ [5].

5. Conclusion

The semi-local convergence of Homeier method of order three is extended using general conditions on $\mathcal{G}'$ and recurrent majorizing sequences.

References

1. I.K. Argyros. On the Newton - Kantorovich hypothesis for solving equations, *J. Comput. Math.*, 169:315-332, 2004.
2. I.K. Argyros. Computational theory of iterative methods. *Series: Studies in Computational Mathematics*, 15, Editors: C.K.Chui and L. Wuytack, Elsevier Publ. Co. New York, USA, 2007.
3. I.K. Argyros. *Convergence and Applications of Newton-type Iterations*. Springer Verlag,Berlin, Germany, 2008.
4. I.K. Argyros and S. Hilout. Weaker conditions for the convergence of Newton's method. *J. Complexity,* 28:64-387, 2012.
5. I.K. Argyros. Unified convergence criteria for iterative Banach space valued methods with applications. *Mathematics*, 9:1942, 2021.
6. I.K. Argyros and A. A. Magréñan. *Iterative methods and their dynamics with applications*. CRC Press, New York, USA, 2017.
7. I.K. Argyros and A. A. Magréñan. *A contemporary study of iterative methods*. Elsevier (Academic Press), New York, 2018.
8. R. Behl, P. Maroju, E. Martinez and S. Singh. A study of the local convergence of a fifth order iterative method. *Indian J. Pure Appl. Math.*, 51(2):439-455, 2020.
9. E. Cătinaş. The inexact, inexact perturbed, and quasi-Newton methods are equivalent models. *Math. Comp.*, 74:291-301, 2005.
10. J.A. Ezquerro, J. M. Gutiérrez, M. A. Hernández, N. Romero and M.J. Rubio. The Newton method: from Newton to Kantorovich (Spanish). *Gac. R. Soc. Mat. Esp.*, 13:53-76, 2010.
11. J. A. Ezquerro, M. A. Hernandez. *Newton's method: An updated approach of Kantorovich's theory*, Cham Switzerland, 2018.
12. M. Grau-Sánchez, A. Grau and M. Noguera. Ostrowski type methods for solving systems of nonlinear equations. *Appl. Math. Comput.*, 281:2377-2385, 2011.
13. H.H.H. Homeier. A modified Newton method with cubic convergence: the multivariate case. *J. Comput. Appl. Math*, 169:161-169, 2004.
14. L.V. Kantorovich, G.P. Akilov. *Functional Analysis*. Pergamon Press, Oxford, 1982.
15. A. A. Magréñan, I. K. Argyros, J. J. Rainer and J. A. Sicilia. Ball convergence of a sixth-order Newton-like method based on means under weak conditions. *J. Math. Chem.*, 56:2117-2131, 2018.
16. A. A. Magréñan and J.M. Gutiérrez. Real dynamics for damped Newton's method applied to cubic polynomials. *J. Comput. Appl. Math.*, 275:527-538, 2015.
17. L.M. Ortega and W. C. Rheinboldt. *Iterative Solution of Nonlinear Equations in Several Variables*. Academic press, New York, 1970.
18. A. M. Ostrowski. *Solution of equations in Euclidean and Banach spaces*, Elsevier, 1973.
19. F. A. Potra and V. Pták. *Nondiscrete induction and iterative processes*. Research Notes in Mathematics, 103. Pitman (Advanced Publishing Program), Boston,MA, 1984.
20. P.D. Proinov. General local convergence theory for a class of iterative processes and its applications to Newton's method, *J. Complexity*, 25:38-62, 2009.
21. P.D. Proinov. New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems. *J. Complexity*, 26:3-42, 2010.
22. W.C. Rheinboldt. *An adaptive continuation process of solving systems of nonlinear equations.* Polish Academy of Science, Banach Ctr. Publ. 3:129-142, 1978.
23. S.M. Shakhno, O. P. Gnatyshyn. On an iterative algorithm of order 1.839... for solving nonlinear least squares problems. *Appl. Math. Applic.*, 161:253-264, 2005.
24. S. M. Shakhno, R. P. Iakymchuk, H. P. Yarmola. Convergence analysis of a two step method for the nonlinear squares problem with decomposition of operator. *J. Numer. Appl. Math.*, 128:82-95, 2018.
25. J.R. Sharma, R.K. Guha and R. Sharma. An efficient fourth order weighted - Newton method for systems of nonlinear equations. *Numer. Algorithms*, 62:307-323, 2013
26. J. F. Traub. *Iterative methods for the solution of equations*. Prentice Hall, New Jersey, USA, 1964.
27. R. Verma. *New Trends in Fractional Programming*. Nova Science Publisher, New York, USA, 2019.

  Comput. Math. Comput. Model. Appl. **2022**, Vol. 1, Iss. 1, pp. 63–68